Instructions

1. The theoretical competition will be 5 hours in duration and is marked out of a total of 300 points.

2. There are **Detailed Worksheets** for carrying out detailed work / rough work. On each of the **Detailed Worksheets**, please fill in

- Student Code
- Question No.
- Page no. and total number of pages.

3. Start each problem on a new page of the Detailed Worksheets. Please write only on the printed side of the sheet. Do not use the reverse side. If you have written something on any sheet which you do not want to be marked, cross it out.

4. There is a summary **Answer Sheet** with your student ID code for your final answers.

5. Please remember that the graders may not understand your language. As far as possible, write your solutions only using mathematical expressions and numbers. If it is necessary to explain something in words, please use short phrases (if possible in English).

6. You are not allowed to leave your exam desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, need more Detailed Worksheets, etc.), please put up your hand to signal the invigilator.

7. The beginning and end of the competition will be indicated by a long sound signal. Additionally, there will be a short sound signal fifteen minutes before the end of the competition (before the final long sound signal).

8. Wait at your table until your envelope is collected. Once all envelopes are collected, your student guide will escort you out of the competition room.

9. A list of constants for this competition is given on the next page.

Fundamental Constants

Speed of light in a vacuum	$c = 2.998 \times 10^{8}\ \mathrm{m\ s^{-1}}$
Planck constant	$h = 6.626 \times 10^{-34}\ \mathrm{J\ s}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	$\sigma = 5.670 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Elementary charge	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
Universal Gravitational constant	$G = 6.674 \times 10^{-11}\ \mathrm{N\ m^{2}\ kg^{-2}}$
Universal gas constant	$R = 8.315\ \mathrm{J\ mol^{-1}\ K^{-1}}$
Avogadro constant	$N_A = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
Wien's displacement law	$\lambda_m T = 2.898 \times 10^{-3}\ \mathrm{m\ K}$
Mass of the electron	$m_e = 9.11 \times 10^{-31}\ \mathrm{kg}$
Mass of the proton	$m_p = 1.67 \times 10^{-27}\ \mathrm{kg}$
Mass of the neutron	$m_n = 1.67 \times 10^{-27}\ \mathrm{kg}$

Astronomical Data

1 parsec	$1\ \mathrm{pc} = 3.086 \times 10^{16}\ \mathrm{m} = 206\,265\ \mathrm{AU} = 3.262\ \mathrm{ly}$
1 Astronomical Unit (AU)	$1\ \mathrm{AU} = 1.496 \times 10^{11}\ \mathrm{m}$
1 Jansky	$1\ \mathrm{Jy} = 10^{-26}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$
Hubble constant	$H_0 = 70\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$
Solar luminosity	$L_\odot = 3.826 \times 10^{26}\ \mathrm{W}$
Apparent angular diameter of Sun	$\theta_\odot = 32'$
Effective temperature of Sun	$T_{\mathrm{eff},\odot} = 5778\ \mathrm{K}$
Obliquity of the Ecliptic (Earth)	$\varepsilon = 23.5^\circ$
Inclination of the lunar orbit w.r.t. Ecliptic	$05^\circ 08' 43''$
Apparent visual magnitude of full moon	-12.74
North Ecliptic Pole (J2000.0)	$(\alpha_E, \delta_E) = (18^{\mathrm{h}}00^{\mathrm{m}}00^{\mathrm{s}}, +66^\circ 33' 39'')$
North Galactic Pole (J2000.0)	$(\alpha_G, \delta_G) = (12^{\mathrm{h}}51^{\mathrm{m}}26^{\mathrm{s}}, +27^\circ 07' 42'')$
1 sidereal day	$23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}}$
1 tropical year	365.2422 solar days
1 sidereal year	365.2564 solar days

Solar magnitudes

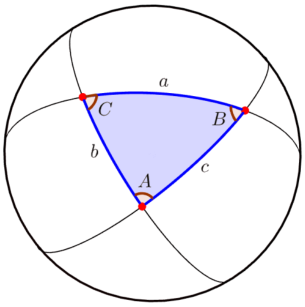
Apparent visual	$= -26.75$
Absolute visual	$= +4.82$
Apparent bolometric	$= -26.83$
Absolute bolometric	$= +4.74$

Solar System

Object	Mean radius (km)	Mass (kg)	Semi-major axis (AU)	Eccentricity	Albedo
Sun	695 500	1.988×10^{30}	---	---	---
Mercury	2 440	3.301×10^{23}	0.387	0.206	0.088
Venus	6 052	4.867×10^{24}	0.723	0.007	0.76
Earth	6 378	5.972×10^{24}	1.000	0.016710	0.31
Moon	1 737	7.346×10^{22}	3.844×10^{5} km	0.054900	0.11
Mars	3 390	6.417×10^{23}	1.524	0.093	0.25
Jupiter	69 911	1.898×10^{27}	5.203	0.048	0.51
Saturn	58 232	5.683×10^{26}	9.537	0.054	0.34
Uranus	25 362	8.681×10^{25}	19.189	0.047	0.30
Neptune	24 622	1.024×10^{26}	30.070	0.009	0.29

Spherical trigonometry

Spherical Cosine theorem	$\cos a = \cos b \cos c + \sin b \sin c \cos A$
Spherical Sine theorem	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$
Four parts formula	$\cot b \sin a = \cos a \cos C + \sin C \cot B$

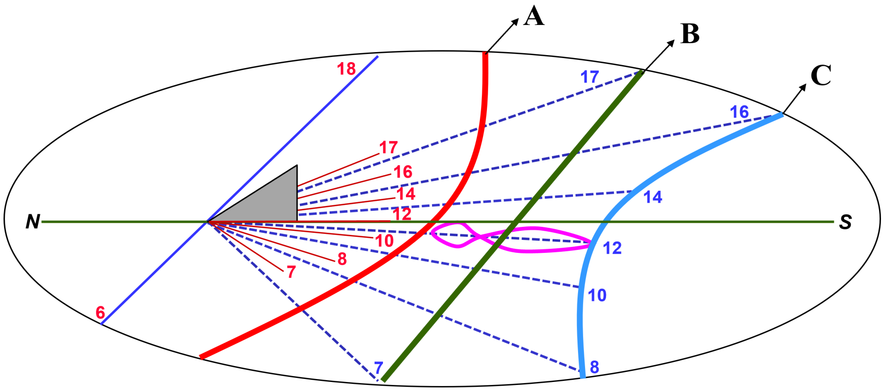


Calculus

Chain rule for derivatives	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
Product rule for derivatives	$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
Power rule for integrals	$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \quad (\text{where } r \neq -1)$
Useful derivatives	$\frac{d\ln(x)}{dx} = \frac{1}{x}$ and $\frac{d\tan(x)}{dx} = \sec^2(x)$

T1. Sundial (10 points)

The following diagram represents a sundial, for which the triangle, called a gnomon, casts a shadow onto the surrounding surface, which has markings and numbers representing important information. It is known that this sundial is located either between the Tropic of Cancer and the Arctic Circle or between the Tropic of Capricorn and the Antarctic Circle.



In the image above, the angle between the dashed and solid lines for any given time is always equal to the longitude difference between the time shown by the sundial and the civil time (the time shown on your watch). For instance, the dashed line corresponding to 7h and the solid line corresponding to 7h form an angle equal to the longitude difference between the location of this sundial and the central meridian of the time zone.

Throughout the year, the shadow of the tip of the gnomon is always between curves A and C.

Read the following statements and indicate whether they are true or false. For each item, write a **T** on the answer sheet if you think the statement is true and an **F** if you think the statement is false. **There is no need to explain your answers.**

(a) This sundial will only function properly if it is located in the southern hemisphere.

(b) Curve **A** represents the trajectory of the shadow of the tip of the gnomon throughout the winter solstice (in the hemisphere the sundial is located in).

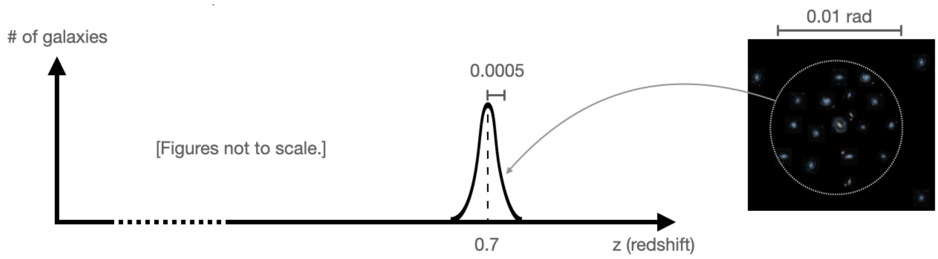
(c) Line **B** represents the trajectory of the tip of the gnomon's shadow throughout the equinoxes.

(d) The solid radial lines provide the mean local solar time.

(e) The analemma shape around the dashed line corresponding to 12h shows the position of the tip of the gnomon's shadow during the true solar noon at the central meridian of the time zone throughout the year.

T2. Galaxy Cluster (10 points)

An astrophysical survey mapped all the galaxies in a small region of the sky, of angular diameter $\Delta\theta = 0.01\ \mathrm{rad}$, where many galaxies seemed to be concentrated around the central area of the image. When the positions and redshifts of all the galaxies in this cluster were measured, an interesting distribution emerged, which is shown in the plot below.



Using these observations, estimate the total mass of the galaxy cluster and express your answer in solar masses. Assume that this galaxy cluster is in dynamical equilibrium, with a root-mean-square redshift dispersion $\sigma_z = \sqrt{\langle (z-0.7)^2 \rangle} = 0.0005$. Feel free to make reasonable approximations when considering the average velocities, masses, and spatial distribution of the galaxies.

Consider that the distance to $\bar{z} = 0.7$ in the standard cosmological model is $D_A = 1500\ \mathrm{Mpc}$. Ignore cosmological effects on the distance.

T3. Asteroid (10 points)

A peculiar asteroid of mass, m, was spotted at a distance, d, from a star with mass, M. The magnitude of the asteroid's velocity at the time of the observation was $v = \sqrt{\frac{GM}{d}}$, where G is the universal gravitational constant. The distance d is much larger than the radius of the star.

For both of the following items, express your answers in terms of M, d, and physical or mathematical constants.

(a) (8 points) If the asteroid is initially moving exactly towards the star, how long will it take for it to collide with the star?

(b) (2 points) If the asteroid is instead initially moving exactly away from the star, how long will it now take for it to collide with the star?

T4. White Dwarf (10 points)

The structure of a white dwarf is sustained against gravitational collapse by the pressure of degenerate electrons, a phenomenon explained by quantum physics and related to the Pauli Exclusion Principle for electrons. The equation of state of a gas made of non-relativistic degenerate electrons is the following:

$$P = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3},$$

where n_e is the number of electrons per unit volume, which can be expressed in terms of the mass density ρ using the dimensionless factor μ_e, the number of nucleons (protons and neutrons) per unit electron. Also consider that the central pressure can be described by this equation of state.

In the condition of hydrostatic equilibrium, the pressure and gravitational forces balance each other at any distance r from the centre of the star. This condition can be expressed by:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2},$$

where $M(r)$ is the mass contained in the sphere of radius r, and $\rho(r)$ is the mass density of the star at a radius r.

Assume that $m_p = m_n$, the density of a white dwarf is roughly uniform, and the following approximation is valid at the surface of the star:

$$\frac{\mathrm{d}P}{\mathrm{d}r}\bigg|_{r=R} \approx -\frac{P_c}{R},$$

where P_c is the pressure at the centre of the star, and R the star radius.

(a) (6 points) The relationship between the mass M and the radius R of a white dwarf can be written in the form:

$$R = a \cdot M^b$$

Find the exponent b and determine the coefficient a in terms of physical constants and μ_e.

(b) (4 points) Using the relationship found in the previous part, estimate the radius of a white dwarf made of fully ionised carbon (${}^{12}_{6}\mathrm{C}$) with a mass of $M = 1.0\,M_\odot$.

T5. CMB (10 points)

The Cosmic Microwave Background (CMB) is radiation emitted during the early Universe. It is reasonably homogeneous and isotropic, and well described by a black-body radiation spectrum. Its emission spectrum today has a peak at a temperature of approximately $T_{\mathrm{today}} \sim 3\ \mathrm{K}$, as measured by the COBE satellite FIRAS instrument.

(a) (3 points) What is the redshift (z) at which the CMB spectrum has a peak at the infrared wavelength of $\lambda_{\mathrm{IR}} \sim 0.1\ \mathrm{mm}$?

(b) (7 points) By assuming a spatially flat, matter-dominated Universe, what is the age of the Universe corresponding to the redshift of the previous part?

T6. Cluster Photography (20 points)

An astronomer takes pictures, in the V-band, of a faint celestial target, from a place with no light pollution. The selected target is the globular cluster Palomar 4, which has an angular diameter of $\theta = 72.0''$ and a uniform surface brightness in the V-band of $m_V = 20.6$ mag /arcsec^2. The observation equipment consists of one reflector telescope, with diameter $D = 305$ mm and f-ratio $f/5$, and a prime focus CCD with quantum efficiency $\eta = 80\%$ and square pixels with size $\ell = 3.80$ μm.

Given data:

- V-band central wavelength: $\lambda_V = 550$ nm
- V-band bandwidth: $\Delta\lambda_V = 88.0$ nm
- Photon flux for a 0-magnitude object in the V-band:
  10000 counts/nm/cm^2/s

(a) (3 points) Calculate the plate scale (the angle of sky projected per unit length of the sensor) of the observation equipment in arcmin/mm.

(b) (4 points) Estimate the number of pixels, n_p, covered by the cluster image on the CCD.

(c) (13 points) With an exposure time of $t = 15$ seconds, the astronomer obtains a signal-to-noise ratio of $S/N = 225$. Compute the brightness of the sky at the observation site, knowing that the CCD has a readout noise (standard deviation) of 5 counts per pixel and dark noise of 6 counts per pixel per minute. Give your answer in mag/arcsec^2. You may find useful: $\sigma^2_{RON} = n_p \cdot 1 \cdot RON^2$ and $\sigma^2_{DN} = n_p \cdot DN \cdot t$.

T7. Castaway (20 points)

After surviving a shipwreck and reaching a small island in the southern hemisphere, a sailor had to estimate the island's latitude using the Sun.

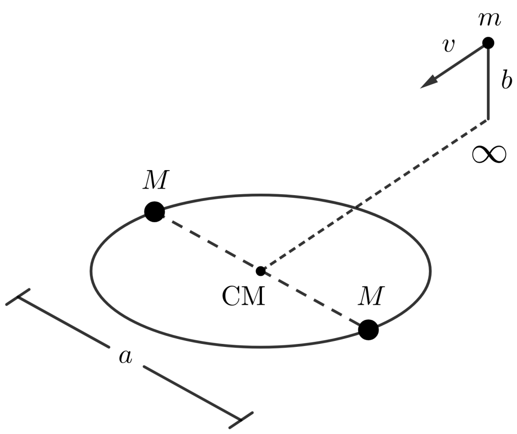
However, due to poor eyesight, the sailor couldn't see the night stars very well, so his best option was to rely on the Sun. He had no information about the date, but he realized the days were longer than the nights.

(a) (7 points) The sailor noticed that on his first day on the island, the angle between the positions of sunrise and sunset on the horizon was 120°. With this piece of information, determine the range of possible values for the latitude of the island. Neglect the daily motion of the Sun across the ecliptic.

(b) (13 points) The angle between the positions of sunrise and sunset kept increasing daily. After 40 days, this angle was equal to 163°. Estimate the latitude of the island. You may neglect the eccentricity of the Earth's orbit.

T8. Binary Hardening (25 points)

Consider a binary system of black holes, both of equal mass M, separated by a distance a, and revolving around their common centre of mass (CM) in circular orbits. This binary system moves against, and interacts with, a very large, uniform field of stars (each of mass $m \ll M$) with number density n. Consider a star that approaches the system from infinity with speed v and impact parameter b, in the reference frame of the CM (as shown in the figure below). Its closest approach distance to the CM is $r_p \approx \frac{1}{2}a$. For tasks (a) and (c), you should make use of the fact that $v^2 \ll \frac{GM}{a}$.



(a) (5 points) Obtain an expression for b, in terms of M, a, v, and physical constants. In this task, assume that the star interacts with the binary as if its total mass was fixed at the CM.

After a complex interaction with the binary, the star is slingshot from the system. The exact calculation of its ejection speed is complex, but the result can be estimated by considering that the star only interacts with one of the components when near the system. As such, consider, in part (b), *only* the gravitational interaction between the star and *one of the components* in the binary.

(b) (6 points) The star approaches the component with an initial speed negligible compared to the component's orbital speed, and both are moving directly towards each other. After interacting with the system, when the star is again far away from the black hole, we find that the direction of its velocity vector is reversed and the final speed is v_f. Determine v_f, in terms of M, a, and physical constants. Assume that linear momentum and mechanical energy are conserved during this interaction and that it takes place in a timescale much smaller than the binary's period. Recall that $m \ll M$.

For the following task, assume that all stars approaching the system from infinity with an impact parameter $\frac{1}{2}b_0 \leq b \leq \frac{3}{2}b_0$ (where b_0 is the impact parameter of a star whose speed at infinity is v_0) attain a closest-approach distance $r_p \approx \frac{1}{2}a$ to the CM. **Also, assume that all stars exit the system with the speed found in (b).**

(c) (14 points) Upon each encounter, part of the total energy of the binary is transferred to the kinetic energy of the star. Assume that the binary orbit remains circular. Knowing this, using your results from previous tasks, and taking into account *only* encounters with the stars within the specified range of impact parameters, show that the reciprocal of the binary's separation increases at a constant rate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = H\frac{G\rho}{v_0}$$

Here, $\rho = nm$ is the mass density of the star field, and G is the universal gravitational constant. Find the dimensionless constant H, which refers to hardening.

T9. Physics of Accretion (35 points)

The accretion of matter onto compact objects, such as neutron stars and black holes, is one of the most efficient ways to produce radiant energy in astrophysical systems. Consider an element of gas of mass Δm in a stationary and geometrically thin disc of matter with a maximum radius of R_{max} and minimum stable orbital radius of R_{min} (with $R_{min}/R_{max} \ll 1$), in rotation around a compact object of mass M and radius R.

(a) (6 points) Assuming that an element of gas in the disc follows an approximately Keplerian circular orbit, find an expression for the total mechanical energy per unit mass $\frac{\Delta E}{\Delta m}$ released by this gas from when it starts orbiting at a radius R_{max} until it reaches an orbital radius of $r \ll R_{max}$. This process occurs very slowly, transforming kinetic energy into internal energy of the gas disc through viscous dissipation. **Note:** Ignore the gravitational interaction between particles within the accretion disc and give your final answer in terms of G, M, and r.

(b) (5 points) Considering that the disc receives mass at an average rate of $\dot{M}$, and assuming that all the mechanical energy lost is ultimately converted into radiation, find an expression for the total luminosity of the disc.

(c) (8 points) Consider now the ring composed of all mass elements with radii between r and $r + \Delta r$. In this scenario, find an expression for the luminosity generated by the disc over its small width Δr at this radius; that is, find the expression for $\frac{\Delta E}{\Delta t \Delta r}$.

(d) (10 points) Assuming that the gravitational energy released in this ring is locally emitted by the surface of the ring in the form of black-body radiation, find an expression for the surface temperature T of the ring.

(e) (3 points) Consider that the central object is a stellar black hole with a mass of $3M_\odot$ and a rate of accretion of $\dot{M} = 10^{-9}\ M_\odot/\mathrm{year}$. Consider also that $R_{min} = 3R_{sch}$, where R_{sch} is the Schwarzschild radius of the black hole. Determine the luminosity of the disc and the peak wavelength of emission of its innermost part. Ignore gravitational redshift effects and assume that the emission from the innermost part of the ring dominates the total emission.

(f) (3 points) Now, considering another accretion system with $\dot{M} = 1\ M_\odot/\ \mathrm{year}$ and a peak emission wavelength of $\lambda = 6 \times 10^{-8}\ \mathrm{m}$, estimate the mass of this black hole.

T10. Greatest Eclipse (75 points)

The greatest eclipse is defined as the instant when the axis of the Moon's shadow cone gets closest to the centre of the Earth in a solar eclipse. This problem explores the geometry of this phenomenon, using the solar eclipse of 29$^{\text{th}}$ May 1919 as an example, as it has great historical significance for being the first time astronomers were able to observationally verify general relativity. One of the scientific expeditions to observe this eclipse took place in the Brazilian city of Sobral.

The two following tables show the Cartesian and spherical coordinates of the Sun and the Moon at the time of the greatest eclipse. The system used for these coordinates is right-handed and has the origin at the centre of the Earth, the positive x-axis pointing towards the Greenwich meridian, and the positive z-axis pointing towards the North Pole. For the rest of this problem, this will be referred to as **System I**.

Spherical Coordinates:

	Centre of the Sun	Centre of the Moon
Radial Distance (r)	$1.516 \times 10^{11}\ \mathrm{m}$	$3.589 \times 10^{8}\ \mathrm{m}$
Polar Angle (θ)	$68^o 29' 44.1''$	$68^o 47' 41.6''$
Azimuthal Angle (φ)	$-1^h 11^m 28.2^s$	$-1^h 11^m 22.9^s$

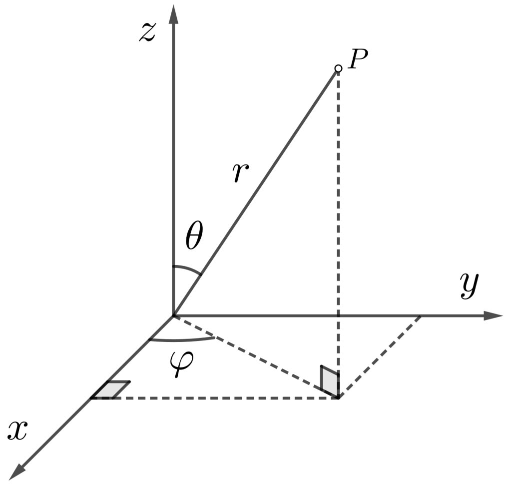
Cartesian Coordinates:

	Centre of the Sun	Centre of the Moon
x	$1.342 \times 10^{11}\ \mathrm{m}$	$3.185 \times 10^{8}\ \mathrm{m}$
y	$-4.327 \times 10^{10}\ \mathrm{m}$	$-1.025 \times 10^{8}\ \mathrm{m}$
z	$5.557 \times 10^{10}\ \mathrm{m}$	$1.298 \times 10^{8}\ \mathrm{m}$

For this problem, assume that the Earth is a perfect sphere.

Note: The spherical coordinates of a point P are defined as follows:

- Radial distance (r): distance between the origin (O) and P (range: $r \geq 0$).
- Polar angle (θ): angle between the positive z-axis and the line segment OP (range: $0^o \leq \theta \leq 180^o$)
- Azimuthal angle (φ): angle between the positive x-axis and the projection of the line segment OP onto the xy-plane (range: $-12^h \leq \varphi < 12^h$)



Part I: Geographic Coordinates (25 points)

(a) (3 points) Determine the declination of the Sun and the Moon during the greatest eclipse for a geocentric observer.

(b) (3 points) Determine the right ascension of the Sun and the Moon at the time of the greatest eclipse for a geocentric observer. The local sidereal time at Greenwich at that same moment was $5^h32^m35.5^s$.

(c) (4 points) Find a unit vector that indicates the direction of the axis of the Moon's shadow cone. This vector should point from the Moon to the vicinity of the centre of the Earth.

(d) (15 points) Determine the latitude and the longitude of the point where the axis of the Moon's shadow cone crosses the surface of the Earth during the greatest eclipse.

Part II: Duration of the Totality (50 points)

Precisely determining the duration of totality of a solar eclipse involves complex calculations that would be beyond the scope of this problem. However, it is possible to obtain a reasonable approximation for this value using the two following assumptions:

- The size of the umbra on the surface of the Earth remains roughly constant throughout totality at a given location.
- The velocity of the umbra on the surface of the Earth remains roughly constant throughout totality at a given location.

(e) (10 points) Estimate the radius of the umbra during the greatest eclipse. In order to simplify the calculations, assume that the umbra is small enough that it can be considered approximately flat and that the axis of the Moon's shadow cone is extremely close to the centre of the Earth during the greatest eclipse.

(f) (3 points) Calculate the velocity of the Earth's rotation at the latitude of the centre of the umbra.

(g) (4 points) Determine the orbital velocity of the Moon at the instant of the greatest eclipse. Neglect the changes in the semi-major axis of the Moon's orbit.

For the remaining items of this problem, assume that the tangential velocity of the Moon is roughly the same as the orbital velocity and neglect its radial component.

In order to calculate the velocity of the umbra, it is convenient to define two new additional right-handed coordinate systems. **System II** is defined as follows:

- Origin (O_{II}): position of the Moon at the instant of the greatest eclipse.
- Positive x-axis: Tangent to the declination circle. Points eastwards.
- Positive y-axis: Tangent to the meridian of right ascension. Points northwards.

System III is defined as follows:

- Origin (O_{III}): centre of the umbra at the instant of the greatest eclipse.
- Positive x-axis: Tangent to the latitude circle. Points eastwards.
- Positive y-axis: Tangent to the meridian of longitude. Points northwards.

Note that in both systems, the xy-plane is tangent to the celestial sphere at the position of the origin.

System III is similar to System II, with the only difference being that the origin (O_{III}) is at the centre of the umbra at the moment of the greatest eclipse.

(h) (14 points) Using System II, determine the velocity vector of the Moon during the greatest eclipse. Note that the intersection between the Celestial Equator and the lunar orbit that is closer to the position of the eclipse has a right ascension of $23^h07^m59.2^s$.

(i) (10 points) Write the velocity vector of the Moon in System III. Note that in System I, the azimuthal angle difference between the positions of the origins O_{II} and O_{III} is negligible, so you should only take into account the difference in the polar angles.

(j) (6 points) Calculate the speed of the centre of the umbra along the surface of the Earth at the instant of the greatest eclipse.

(k) (3 points) Estimate the duration of the totality of the eclipse at the location with the coordinates found in part (d).

T11. Ground Tracks (75 points)

The projection of a satellite's orbit onto the Earth's surface is called its ground track. At a given instant, one can imagine a radial line drawn outward from the centre of the Earth to the satellite. The intersection between the Earth's spherical surface and this radial line is a point on the ground track.

The location of this point is specified by its geocentric latitude and longitude. The ground track is then essentially the figure traced by this point as the satellite moves around the Earth.

Part I: Sun-Synchronous Orbits (25 points)

It is particularly interesting to analyse the ground track of a so-called Sun-Synchronous orbit. This is a nearly polar orbit around a planet where the satellite passes over any given point on the planet's surface at the same mean local solar time. This property is especially interesting for satellite imaging, ensuring similar illumination conditions over different days.

The figure below shows the ground track of a satellite in a Sun-Synchronous orbit. Its inclination angle (*i*) - the angle between the satellite orbital plane and the Earth's equatorial plane - falls within the range $90^\circ < i < 180^\circ$. The graph depicts five complete orbits of the satellite.

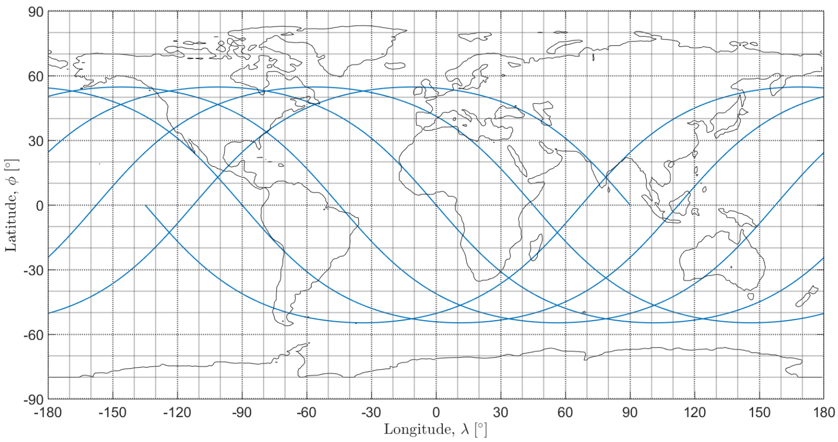


Figure 2 - Ground Track for five orbital periods of the satellite

For the questions in Part I, assume that the Earth's orbit around the Sun is circular.

(a) (3 points) Determine the nodal precession rate for the orbit in rad/s.

(b) (8 points) Based on the ground track shown in Figure 2, determine the inclination of the satellite's orbit (in degrees) and estimate its orbital period (in minutes). Consider that the orbital period of the satellite is shorter than one sidereal day.

(c) (2 points) Calculate the semi-major axis *a* of the orbit in km.

(d) (1 point) Determine the number of orbits completed by the satellite until it returns to the same position on Earth.

(e) (11 points) As seen in Figure 2, the ground track crosses the Brazilian city of Maceió $(\phi, \lambda) = (9.7^\circ S; 35.7^\circ W)$ and also Chorzów $(\phi, \lambda) = (50.3^\circ N; 19.0^\circ E)$, in Poland. Knowing that the ground track crosses Maceió at noon (local time), determine the local time that the satellite track crosses Chorzów. Hint: specifically for this task, you may neglect the effects of nodal precession.

Part II: Tundra orbits (50 points)

A Tundra orbit is a type of geosynchronous elliptical orbit characterised by a high inclination. The apogee is positioned over a specific geographic region, allowing for prolonged visibility and coverage over that area. This orbit ensures that a satellite spends the majority of its orbital period over the northern - or southern - hemisphere, making it particularly useful for communications and weather observation over high-latitude regions.

The image below represents the ground track of a satellite in a Tundra orbit with an argument of perigee equal to 270°. The satellite orbits the Earth in the same direction as its rotation. For the following items, you can ignore the effects of the Earth's oblateness.

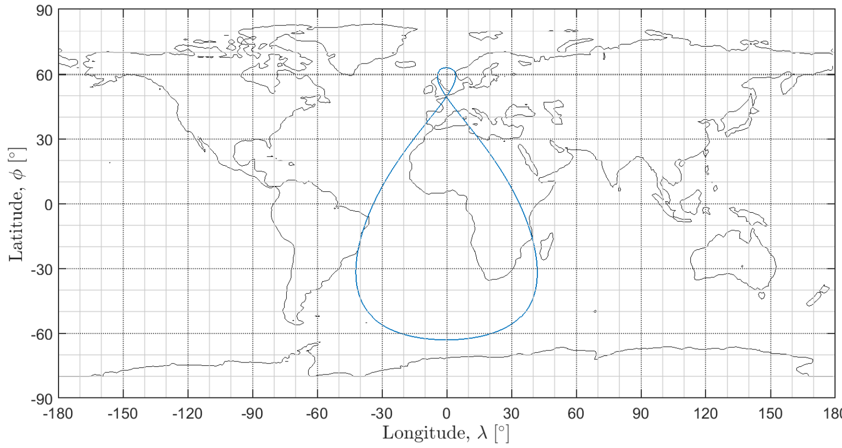


Figure 3 - Tundra Orbit Ground Track for one orbital period of the satellite

(f) (4 points) Based on the graph above, give the inclination of the satellite's orbit *i* (in degrees), its orbital period *T* (in minutes), and its semi-major axis *a* (in km).

(g) (12 points) Show that the time a satellite spends in the northern hemisphere is given by

$$T' = \left(\frac{1}{2} + \frac{\sin^{-1}(e)}{\pi} + \frac{e}{\pi} \cdot \sqrt{1-e^2}\right)T$$

where *e* is the eccentricity of the orbit and *T* is its orbital period.

(h) (10 points) Estimate numerically the eccentricity e of its orbit. You can consider that the eccentricity is so small that $\sin(e) \approx e$ and $e^2 \ll 1$.

(i) (18 points) From the ground track, we can observe that the satellite exhibits retrograde motion in both its northern and southern hemisphere trajectories. Find the true anomaly (in degrees) of the satellite at the beginning and end of its retrograde motion in the southern hemisphere.

(j) (6 points) It is also noticeable that the ground track of a Tundra orbit has the shape of a figure-8, similar to an analemma, so that the satellite passes over the same point on Earth in a single orbit. Calculate the minimum eccentricity the orbit would need to have for this property to cease occurring. Use the same orbital inclination as the orbit in Figure 3.