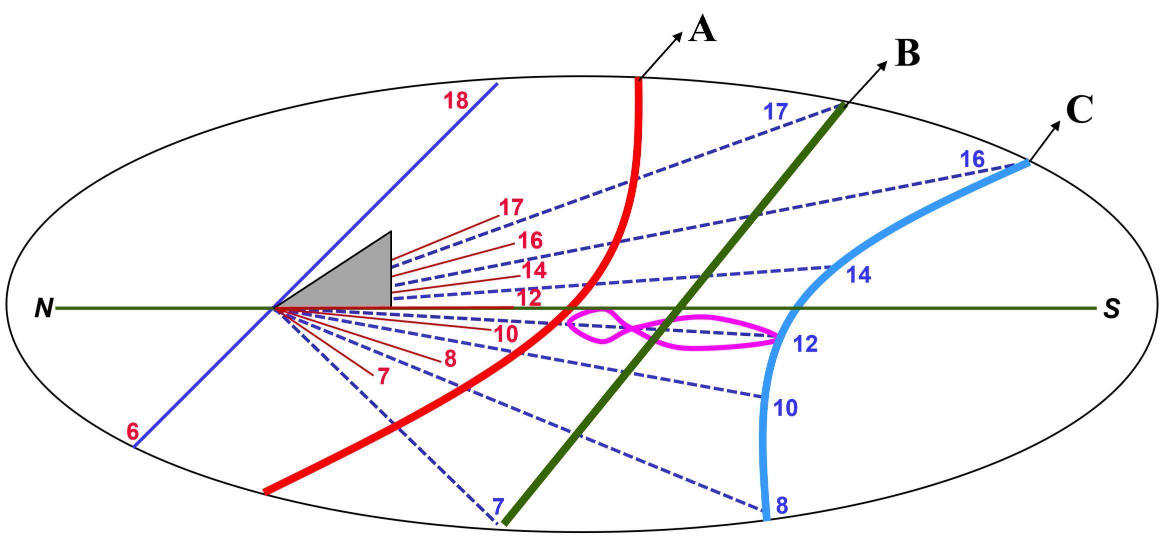
Short Questions

(T1) Sundial (10 points)

The following diagram represents a sundial, where the triangle, named Gnomon, casts a shadow on the surface, where there are markings and numbers representing important information. It is known that this sundial is located either between the Tropic of Cancer and the Artic Circle or between the Tropic of Capricorn and the Antartic Circle.



In the image above, the angle between the dashed and solid lines for any given time is always equal to the longitude difference between the time shown by the sundial and the civil time (time in your watch). For instance, the dashed line corresponding to 7h and the solid line corresponding to 7h form an angle equal to the longitude difference between the location of this sundial and the central meridian of the timezone.

Throughout the year, the shadow of the tip of the gnomon is always between curves A and C.

Read the following statements and indicate whether they are true or false. For each item, write a "**T**" in the answer sheet if you think the statement is true and an "**F**" if you think the statement is false. **There is no need to explain your answers.**

(a) This sundial will only function properly if it is located in the southern hemisphere.

(b) Curve A represents the trajectory of the shadow of the tip of the gnomon throughout the winter solstice day of the hemisphere where the sundial is located.

(c) Line B represents the trajectory of the tip of the gnomon's shadow throughout the equinoxes.

(d) The solid radial lines provide the mean local solar time.

(e) The analemma shape around the dashed line corresponding to 12h shows the position of the tip of the gnomon's shadow during the true solar noon at the central meridian of the time zone throughout the year.

Solution:

(a) **True**. The polar rod of the sundial faces the hemisphere that contains the south cardinal point as the fundamental pole, so the sundial was designed for a location in the southern hemisphere of the Earth.

(b) **False**. This curve represents the trajectory of the tip of the rod's shadow throughout the day of the summer solstice in the hemisphere in which the sundial is located.

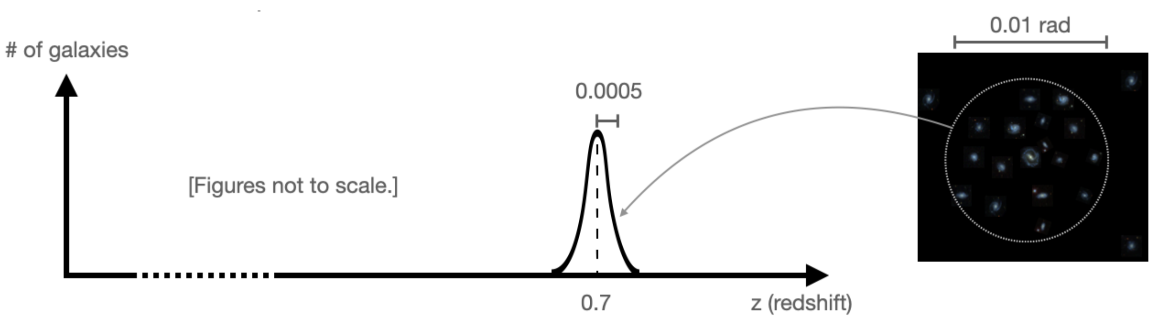
(c) **True**. This straight line, which is perpendicular to the meridian line, represents the trajectory of the tip of the rod's shadow throughout the day of either southern autumnal or vernal equinox.

(d) **False**. This set of radial lines corresponds to the true local solar time. The 12h line coincides with north-south line, which must be the case for the true solar noon.

(e) **True**. The dashed lines and the solid lines form an angle corresponding to the longitude difference between the sundial and the center of the time zone. Therefore, the dashed lines correspond to the true solar time of the center of the time zone. An analemma shaped figure around the line that corresponds to a true local solar time indicates the mean solar time.

(T2) Galaxy Cluster (10 points)

An astrophysical survey mapped all the galaxies in a small region of the sky, of angular diameter $\Delta\theta = 0.01\,\text{rad}$, where many galaxies seemed to be concentrated around the central area of the image. When the positions and redshifts of all the galaxies in this cluster were measured, an interesting distribution emerged, which is shown in the plot below.



Using these observations, estimate the total mass of the galaxy cluster and express your answer in solar masses. Assume that this galaxy cluster is in dynamical equilibrium, with a root-mean-square redshift dispersion $\sigma_z = \sqrt{\langle (z-0.7)^2 \rangle} = 0.0005$. Feel free to make reasonable approximations when considering the average velocities, masses and spatial distribution of galaxies.

Consider that the distance to $\bar{z} = 0.7$ in the standard cosmological model is $D_A = 1500$ Mpc. Disconsider cosmological effects on the distance.

Solution:

If U and K are the total gravitational potential energy and the total kinetic energy of the cluster, respectively, the virial theorem says that:

$$U + 2K = 0$$ **2.0**

Let's say that there are N galaxies of masses m_i, where $i = 1, ..., N$, moving with velocities $\vec{u}_i = \vec{v}_i - \vec{v}_0$ with respect to the cluster, where $\vec{v}_0$ is the velocity of the cluster as a whole. Therefore, in the cluster frame, K is given by

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

As an estimation, consider that all galaxies have mass equal to the average mass $\langle m \rangle = M_T/N$:

$$K = \frac{1}{2}\frac{M_T}{N}\sum_{i=1}^{N} u_i^2 = \frac{1}{2} M_T \frac{\sum u_i^2}{N} = \frac{1}{2} M_T \sigma_v^2$$ **2.0**

Where $\sigma_v = \sqrt{\langle u_i^2 \rangle}$ is the root-mean-square galaxy speed, also known as velocity dispersion. Notice, however, that only the dispersion in redshift is given, which translates into a radial velocity dispersion:

$$\sigma_{v_r} = c \cdot \sigma_z = 2.998 \times 10^8 \text{ m/s} \times 0.0005 = 1.499 \times 10^5 \text{ m/s}$$ **1.0**

There is no penalization for students who use that $\sigma_v = c\sigma_z$. For completeness of the solution, however, we relate σ_v and σ_{v_r} by assuming three-dimensional isotropy for velocities, which yields $\sigma_v^2 = 3\sigma_{v_r}^2$. K is then given by:

$$K = \frac{3}{2} M_T \sigma_{v_r}^2$$

The gravitational potential energy, on the other hand, is given by:

$$U = -\sum_{i \neq j} \frac{G m_i m_j}{r_{ij}}$$

This can be estimated in different ways. Usually, the spatial distribution of the large number of galaxies may be approximated as uniform, so that one estimates U as homogeneous spherical mass distribution, for which U is given by:

$$U = -\frac{3}{5}\frac{G M_T^2}{R}$$ **2.0**

And, assuming the cluster is spherical, its radius is:

$$R = \frac{D_A \cdot \Delta\theta}{2} = \frac{1500 \times 10^6 \times 206265 \times 1.496 \times 10^{11} \times 10^{-2}}{2} = 2.3143 \times 10^{23} \text{ m}$$ **1.0**

Now, using the virial theorem, it is possible to calculate the total mass:

$$\therefore -\frac{3}{5}GM_T^2\frac{1}{R} + 2\cdot\frac{3}{2}M_T\sigma_{v_r}^2 = 0$$
$$\Rightarrow M_T = \frac{5R\sigma_{v_r^2}}{G}$$
$$M_T = \frac{5\times 3.857\times 10^{23}\times(1.499\times 10^5)^2}{6.67\times 10^{-11}}\ \text{kg}$$
$$\boxed{M_T \approx 3.9\times 10^{44}\ \text{kg} = 2.0\times 10^{14}\ \text{M}_\odot}$$ **2.0**

Appendix on calculation of U: Other valid methods for estimating U include explicitly writing the sum as:

$$U = -\frac{1}{2}G\left(\sum_i m_i\right)^2\langle\frac{1}{r_{ij}}\rangle$$
$$U = -\frac{1}{2}\frac{GM_T^2}{R}$$

where the factor of $1/2$ comes from counting all the unique pairs, and $\langle r_{ij}^{-1}\rangle$ was taken (as an approximation) as R^{-1}.

Or, alternatively, using dimensional analysis to argue that:

$$U = -\frac{GM_T^2}{R}$$

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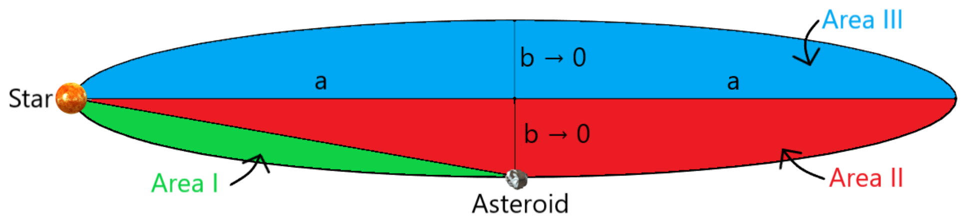
(T3) Asteroid (10 points)

A peculiar asteroid of mass m was spotted at a distance d, from a star with mass M. The modulus of the asteroid's velocity at the time of the observation was $v = \sqrt{\frac{GM}{d}}$, where G is the universal gravitational constant. The distance d is much larger than the radius of the star.

For both of the following items, express your answers in terms of M, d, and physical or mathematical constants.

(a) (8 points) If the asteroid is initially moving exactly towards the star, how long will it take for it to collide with the star?

Solution: In this scenario, the asteroid would fall directly towards the star. However, in order to simplify the calculations, it is possible to consider that the asteroid would be in a degenerate elliptical orbit. In that case, the semi-minor axis would be infinitesimally small and the asteroid would still practically be moving on a straight line. It is also important to notice that the focii would virtually be at the periapsis and apapsis points. The degenerate elliptical orbit is shown in the following figure: **2.0**



The semi-major axis of this elliptical orbit can be obtained through the following expression, where m is the mass of the asteroid:

$$E = \frac{mv^2}{2} - \frac{GMm}{d} = -\frac{GMm}{2a}$$
$$\frac{GMm}{2d} - \frac{GMm}{d} = -\frac{GMm}{2a}$$
$$\frac{1}{2d} = \frac{1}{2a}$$
$$\therefore d = a$$ **2.0**

Now, it is possible to use Kepler's third law to find an expression for the period of the orbit:

$$\frac{T^2}{d^3} = \frac{4\pi^2}{GM} \longrightarrow T = 2\pi d\sqrt{\frac{d}{GM}}$$ **1.0**

If the asteroid is initially moving towards the star, it sweeps area I to reach the star. Using Kepler's second law, it is possible to calculate how long it takes for that to happen:

$$\frac{\Delta t}{T} = \frac{A_I}{A_{Total}} = \frac{\frac{\pi ab}{4} - \frac{ab}{2}}{\pi ab}$$ **2.0**

$$\Delta t = \left(\frac{1}{4} - \frac{1}{2\pi}\right) T$$

$$\boxed{\Delta t = \left(\frac{\pi}{2} - 1\right) d\sqrt{\frac{d}{GM}}}$$ **1.0**

Note that the radius of the star is negligible in these calculations since it is significantly smaller than d.

(b) (2 points) If the asteroid is initially moving exactly away from the star, how long will it take for it to collide with the star?

Solution: If the asteroid is initially moving away from the star, it sweeps areas II and III before reaching the star. It is possible to again use Kepler's second law to find the time interval:

$$\frac{\Delta t}{T} = \frac{A_{II} + A_{III}}{A_{Total}} = \frac{\frac{ab}{2} + \frac{\pi ab}{4} + \frac{\pi ab}{2}}{\pi ab}$$

$$\Delta t = \left(\frac{3}{4} + \frac{1}{2\pi}\right) T$$ **1.5**

$$\boxed{\Delta t = \left(\frac{3\pi}{2} + 1\right) d\sqrt{\frac{d}{GM}}}$$ **0.5**

(T4) White Dwarf (10 points)

The structure of a white dwarf is sustained against gravitational collapse by the pressure of degenerate electrons, a phenomenon explained by quantum physics and related to the Pauli Exclusion Principle for electrons. The equation of state of a gas made of non-relativistic degenerate electrons is the following:

$$P = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3},$$

where n_e is the number of electrons per unit volume, which can be expressed in terms of the mass density ρ of the white dwarf using the dimensionless factor μ_e, the number of nucleons (protons and neutrons) per unit electron. Also consider that the central pressure can be described by this equation of state.

In the condition of hydrostatic equilibrium, the pressure and gravitational forces balance each other at any distance r from the center of the star. This condition can be expressed by:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2},$$

where $M(r)$ is the mass contained in the sphere of radius r, and $\rho(r)$ is the mass density of the star at a radius r.

Assume that $m_p = m_n$ and that the density of a white dwarf is roughly uniform and the following approximation is valid at the surface of the star:

$$\left.\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{r=R} \approx -\frac{P_c}{R},$$

where P_c is the pressure at the center of the star, and R the star radius.

(a) (6 points) The relationship between the mass M and the radius R of a white dwarf can be written in the form

$$R = a \cdot M^b$$

Find the exponent b and determine the coefficient a in terms of physical constants and μ_e.

Solution: Simplifying the expression given in the problem statement:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2}$$

At $r = R$, is is possible use the approximation provided and the expression $\rho = 3M/(4\pi R^3)$:

$$-\frac{P_c}{R} = -\frac{GM\{3M/(4\pi R^3)\}}{R^2}$$
$$P_c = \frac{3GM^2}{4\pi R^4}$$ **1.5**

Furthermore, electron density n_e can be related to mass density as:

$$\rho = \mu_e m_p n_e$$ **1.5**

Using now the equation of state:

$$\begin{aligned} P_c &= \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \left(\frac{\rho}{\mu_e m_p}\right)^{5/3} \\ &= \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \left(\frac{3M}{4\pi R^3 \mu_e m_p}\right)^{5/3} \\ \frac{3GM^2}{4\pi R^4} &= \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \left(\frac{3}{4\pi \mu_e m_p}\right)^{5/3} \frac{M^{5/3}}{R^5} \\ R &= \left(\frac{4\pi}{3}\right) \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5Gm_e} \left(\frac{3}{4\pi \mu_e m_p}\right)^{5/3} M^{-1/3} \end{aligned}$$

Therefore:

$$\boxed{a = \left(\frac{4\pi}{3}\right) \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5Gm_e} \left(\frac{3}{4\pi \mu_e m_p}\right)^{5/3}}$$ **1.5**

$$\boxed{b = -\frac{1}{3}}$$ **1.5**

(b) (4 points) Using the relationship found in the previous item, estimate the radius of a white dwarf made of fully ionized carbon ($^{12}_{6}$C) with a mass of $M = 1.0\, M_\odot$.

Solution: There are 2 nucleons (1 proton and 1 neutron) per unit electron for carbon, so that $\mu_e = 2$. **1.0**

Using the expression for the radius found on the previous item:

$$
\begin{aligned}
R &\approx \left(\frac{4\pi}{3}\right)\left(\frac{3}{8\pi}\right)^{2/3}\frac{1}{5}\left(\frac{3}{4\pi}\right)^{5/3}\frac{h^2}{Gm_e}\frac{1}{(\mu_e m_p)^{5/3}}\frac{1}{M^{1/3}} \\
&\approx 1.866\cdot 10^{-2}\frac{h^2}{Gm_e}\frac{1}{(\mu_e m_p)^{5/3}}\frac{1}{M^{1/3}} \\
&\approx \frac{1.866\times 10^{-2}\times\left(6.626\times 10^{-34}\right)^2}{6.67\times 10^{-11}\times 9.11\times 10^{-31}\times\left(2\times 1.67\times 10^{-27}\right)^{5/3}\times\left(1.988\times 10^{30}\right)^{1/3}}
\end{aligned}
$$

$$\boxed{R \approx 1.44\times 10^6\,\mathrm{m}}$$ **3.0**

(T5) CMB (10 points)

The Cosmic Microwave Background (CMB) is a radiation coming from the early Universe, it is reasonably homogeneous and isotropic and described by a black-body radiation spectrum. Its emission spectrum today has a peak at a temperature of approximately $T_{\text{today}} \sim 3\,\mathrm{K}$, given by COBE satellite FIRAS instrument measurements.

(a) (3 points) What is the redshift (z) at which the CMB spectrum had a peak at the infrared wavelength of $\lambda_{\text{IR}} \sim 0.1\,\mathrm{mm}$?

Solution:
In order to find the redshift, we need λ_{today}. Since T_{today} refers to the temperature at the emission peak, we use Wien's law:

$$\lambda_{\text{today}}\cdot T_{\text{today}} = b$$

$$\lambda_{\text{today}} \approx 1\,\mathrm{mm}$$ **1.0**

So the redshift of the CMB, when it is emmited at the infrared spectrum (with the values mentioned) is

$$z = \frac{\lambda_{\text{today}} - \lambda_{\text{infrared}}}{\lambda_{\text{infrared}}}$$ **1.0**

$$\boxed{z \approx \frac{1-0.1}{0.1} = 9}$$ **1.0**

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Note: It is also possible to find T_{infrared} using Wien's law, and then use the definition of redshift to find z, which leads to the same answer.

(b) (7 points) By assuming a spatially flat matter-dominated Universe, what is the age of the Universe corresponding to the redshift of the previous part?

Solution:
As the Universe is assumed as composed of just matter (i.e. $\Omega_m = 1$), so the expression

for the age of the Universe is given, in terms of the scale factor $\left(a = \frac{1}{1+z}\right)$, by

$$a_{\text{ir}} = \frac{1}{1+z} = \frac{1}{1+9} = 0.1 \qquad \mathbf{1.0}$$

$$\text{Now, } \frac{\mathrm{d}a}{\mathrm{d}t} = H(t) \cdot a(t) \qquad \mathbf{1.0}$$

And, from the first Friedmann equation for a matter-dominated universe, we know that:

$$H^2 = \frac{H_0^2}{a^3} \qquad \mathbf{2.0}$$

Therefore:

$$\frac{da}{dt} = \frac{H_0}{a^{1.5}(t)} a(t) = \frac{H_0}{\sqrt{a(t)}}$$

$$\therefore t = \int_0^{0.1} \frac{\sqrt{a}}{H_0} \mathrm{d}a$$

$$= \frac{1}{H_0} \int_0^{0.1} \sqrt{a}\, \mathrm{d}a = \frac{2}{3\, H_0} a^{1.5}|_0^{0.1} \qquad \mathbf{2.0}$$

$$= \frac{2 \times 0.1^{1.5}}{3 \times 70}$$

$$= 3 \times 10^{-4}\, \mathrm{s\,Mpc/km}$$

$$= \frac{3 \times 10^{-4} \times 3.086 \times 10^{22}}{1000 \times 86400 \times 365.2422}$$

$$\boxed{t \approx 0.3\,\mathrm{Gyr}} \qquad \mathbf{1.0}$$

Medium Questions

(T6) Cluster Photography (20 points)

An astronomer takes pictures, in the V-band, of a faint celestial target, from a place with no light pollution. The selected target is the globular cluster Palomar 6, which has an angular diameter of $\theta = 72.0''$ and a uniform surface brightness in the V-band of $m_V = 20.6\,\mathrm{mag/arcsec^2}$. The observation equipment consists of one reflector telescope, with diameter $D = 305\,\mathrm{mm}$ and F-ratio $f/5$, and a prime focus CCD with quantum efficiency $\eta = 80\%$ and square pixels with size $\ell = 3.80\,\mu\mathrm{m}$.

Given data:

- V-band central wavelength: $\lambda_V = 550\,\mathrm{nm}$
- V-band bandwidth: $\Delta\lambda_V = 88.0\,\mathrm{nm}$
- Photons flux for a 0-magnitude object in the V-band:10 000 counts/nm/cm^2/s)

(a) (3 points) Calculate the plate scale, the angle of sky projected per unit length of the sensor, of the observation equipment in arcmin/mm.

Solution:
The plate scale is simply the angular size of the image per unit length as projected at the focal plane. Hence, for small angles,

$$\tan\theta_i = \frac{\ell_i}{f} \approx \theta_i$$ **1.0**

where θ_i is the angular size of the image, ℓ_i is the unit length, and f is the focal length of the telescope, which is given by (D × F-ratio). **0.5**

$$\theta_i = \frac{1\,\text{mm}}{5 \times 305\,\text{mm}} = 6.56 \times 10^{-4}\,\text{rad} = 2.254'$$
$$\boxed{PS \approx 2.25\,\text{arcmin/mm}}$$ **1.5**

(b) (4 points) Estimate the number of pixels n_p covered by the cluster image on the CCD.

Solution:
Given the plate scale, the diameter of the image on the focal plane is

$$d_{GC} = \frac{72.0''}{2.25' \times 60} = 0.532\,\text{mm}$$ **1.0**

This implies that the area covered by the image is

$$S_{GC} = \frac{\pi}{4} \cdot d_{GC}^2 = 0.223\,\text{mm}^2$$

which can be divided by the area of a single pixel to estimate the pixel coverage

$$n_p = \frac{0.223}{(3.8 \times 10^{-3})^2} \approx 15\,400\,\text{pixels}$$ **2.0**

Since there are other methods to estimate such quantity, the accepted range is

$$15300 \leqslant n_p \leqslant 15600$$ **1.0**

(c) (13 points) With an exposure time of t = 15 s, the astronomer obtains a signal-to-noise ratio of $S/N = 225$. Compute the brightness of the sky at the observation site, knowing that the CCD has a readout noise (standard deviation) of 5 counts/pixel and dark noise of 6 counts/pixel/minute. Give your answer in mag/arcsec^2. You may find useful: $\sigma_{RON}^2 = n_p \cdot 1 \cdot RON^2$ and $\sigma_{DN}^2 = n_p \cdot DN \cdot t$.

Solution: The signal-to-noise ratio is given by the expression

$$S/N = \frac{N_{GC}}{\sqrt{\sigma_{GC}^2 + \sigma_{sky}^2 + \sigma_{RON}^2 + \sigma_{DN}^2}}$$
$$= \frac{N_{GC}}{\sqrt{N_{GC} + N_{sky} + n_p \cdot RON^2 + n_p \cdot DN \cdot t}}$$ **3.0**

where,

$$\begin{cases} N_{GC} \rightarrow \text{ Total Source Count} \\ \sigma_{GC} = \sqrt{N_{GC}} \rightarrow \text{ Poisson Noise} \\ N_{sky} \rightarrow \text{ Total Sky Count} \\ \sigma_{sky} = \sqrt{N_{sky}} \rightarrow \text{ Sky Noise} \\ \sigma_{RON} = \sqrt{n_p \cdot 1 \cdot RON^2} \rightarrow \text{ Readout Noise} \\ \sigma_{DN} = \sqrt{n_p \cdot DN \cdot t} \rightarrow \text{ Dark Noise} \end{cases}$$

The noise values associated with the CCD operation can be easily calculated:

$$\begin{cases} \sigma^2_{RON} = 15400 \times 5^2 = 385000 \text{ counts} \\ \sigma^2_{DN} = 15400 \times 6 \times \dfrac{1}{60} \times 15 = 23100 \text{ counts} \end{cases}$$ **2.0**

Now, the apparent visual magnitude of the globular cluster must be calculated,

$$V_{GC} - m_V = -2.5 \log \left(\frac{\Omega_{GC}}{1} \right)$$

with Ω_{GC} being the solid angle subtended by the globular cluster,

$$\Omega_{GC} = \pi \cdot \left(\frac{\theta}{2} \right)^2 = \pi \times \left(\frac{72.0''}{2} \right)^2 = 4071.5 \, \text{arcsec}^2$$ **1.0**

Hence,

$$V_{GC} = 20.6 - 2.5 \log (4071.5) = 11.6 \, \text{mag}$$ **1.0**

and the associated flux of photons can now be evaluated using the 0-magnitude as a reference:

$$V_{GC} - V_0 = -2.5 \log \left(\frac{\phi_{GC}}{\phi_0} \right)$$

$$\begin{aligned} \phi_{GC} &= \phi_0 \cdot 10^{-V_{GC}/2.5} \\ &\approx 0.234 \, \text{counts/nm/cm}^2\text{/s} \end{aligned}$$ **1.0**

With the calculated parameters, the total source count is:

$$\begin{aligned} N_{GC} &= \eta \cdot \phi_{GC} \cdot \frac{\pi}{4} \cdot D^2 \cdot \Delta\lambda \cdot t \\ N_{GC} &= 0.80 \times 0.234 \times \frac{\pi}{4} \times (30.5)^2 \times 88.0 \times 15 \\ &= 180\,540 \, \text{counts} \end{aligned}$$ **1.0**

By referring to the signal-to-noise ratio expression and substituting the values,

$$\begin{aligned} 225 &= \frac{180540}{\sqrt{180540 + N_{sky} + 385000 + 23100}} \\ N_{sky} &= \left(\frac{180540}{225} \right)^2 - 385000 - 180540 - 23100 \\ N_{sky} &= 55250 \text{ counts} \end{aligned}$$ **1.0**

The inverse procedure to determine the Total Source Count is taken:

$$\phi_{sky} = \frac{N_{sky}}{\eta \cdot \frac{\pi}{4} \cdot D^2 \cdot \Delta\lambda \cdot t}$$
$$= \frac{55250}{0.80 \times \frac{\pi}{4} \times 30.5^2 \times 88.0 \times 15}$$
$$\phi_{sky} = 7.16 \times 10^{-2}\,\text{counts/nm/cm}^2\text{/s} \qquad \textbf{1.0}$$

which implies

$$V_{sky} = V_0 - 2.5 \log\left(\frac{\phi_{sky}}{\phi_0}\right) = 12.9\,\text{mag} \qquad \textbf{1.0}$$

and finally, since $\Omega_{sky} = \Omega_{GC}$

$$m_{sky} = V_{sky} + 2.5 \log\left(\frac{\Omega_{sky}}{1}\right)$$
$$= 12.8 + 2.5 \log(4071.5)$$
$$m_{sky} = 21.9\,\text{mag/arcsec}^2 \qquad \textbf{1.0}$$

Considering the accepted range of pixels from the previous item, the accepted answers for the brightness of the sky fall within the interval

$$21.7 \leqslant m_{sky} \leqslant 22.1\text{mag/arcsec}^2$$

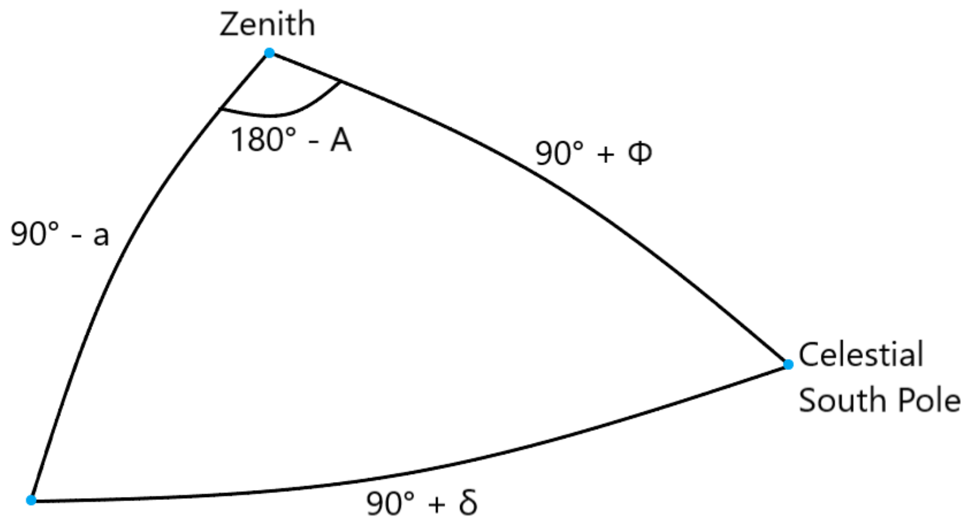
(T7) Castaway (20 points)

After surviving a shipwreck and reaching a small island in the southern hemisphere, a sailor had to estimate the island's latitude using the Sun.

However, due to a poor eyesight, the sailor couldn't see the night stars very well, so his best option was to rely on the Sun. He had no information about the date, but he realized the days were longer than the nights.

(a) **(7 points)** The sailor noticed that on his first day on the island, the angle between the positions of the sunrise and the sunset on the horizon was 120°. With this piece of information, determine the range of possible values for the latitude of the island. Neglect the daily motion of the Sun across the ecliptic.

Solution:
The first step is to find an expression for the azimuth of an object during sunrise:



Using the law of cosines:

$$\cos(90^\circ + \delta) = \cos(90^\circ - a)\cos(90^\circ + \phi) + \sin(90^\circ - a)\sin(90^\circ + \phi)\cos(180^\circ - A)$$
$$\therefore -\sin(\delta) = -\sin(a)\cos(\phi) - \cos(a)\cos(\phi)\cos(A)$$

Since the altitude $a = 0^\circ$ at the sunrise:

$$\cos(\phi) = \frac{\sin(\delta)}{\cos(A)}$$ **2.0**

Note that due to symmetry over the local meridian, the azimuth of sunrise corresponds to 180° minus half of the angle given in the question statement (the 180° term comes from the fact that the observer is on the southern hemisphere).

The scenarios that lead to the minimum and maximum values of latitude correspond to the extreme values for the declination of the Sun in the southern hemisphere summer (-23.5° and 0°).

If the declination of the Sun is -23.5°, the latitude has to be

$$\cos(\phi) = \frac{\sin(-23.5^\circ)}{\cos\left(180^\circ - \frac{120^\circ}{2}\right)}$$
$$\cos(\phi) = -2\ \sin(-23.5^\circ)$$
$$\phi = -37.1^\circ$$ **2.0**

Only the negative solution of this equation is relevant in this case since the observer is in the southern hemisphere.

If the declination is the Sun is 0°, simply plugging in this value to the formula will result in a latitude of -90°. Since celestial objects do not rise or set at the poles, this approach would not be conceptually correct. However, at a point infinitesimally close to the pole, the Sun would still mathematically rise and set, and it would be possible to achieve the difference in azimuth from the problem statement with a declination very

close to 0°. Thus, although a latitude of −90° is impossible, any latitude infinitesimally close to it would still mathematically be possible.
Therefore, the range of possible latitudes for the island is the following:

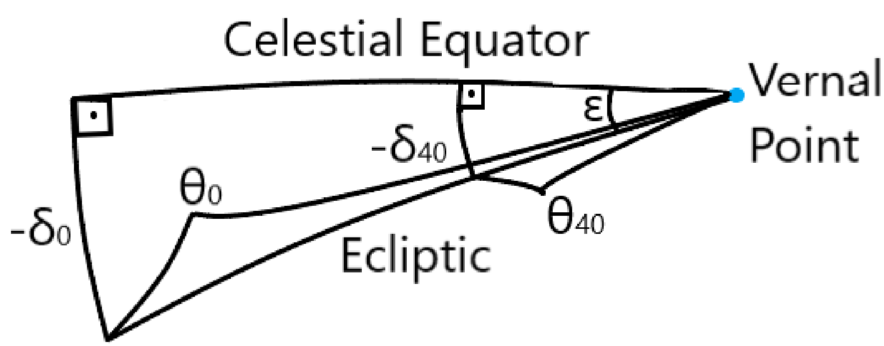
$$\boxed{-90^\circ < \phi \leqslant -37.1^\circ}$$ **3.0**

(b) **(13 points)** The angle between positions of the sunrise and the sunset kept increasing daily. After 40 days, this angle was equal to 163°. Estimate the latitude of the island. You may neglect the eccentricity of the Earth's orbit.

Solution:
Since the latitude is constant, the following expression must be true:

$$\cos(\phi) = \frac{\sin(\delta_0)}{\cos(A_0)} = \frac{\sin(\delta_{40})}{\cos(A_{40})}$$ **2.0**

It is also possible to apply the law of sines to the following triangle to derive another expression for the declination of the Sun on each day:



1.0

$$\frac{\sin(\theta_0)}{\sin(90^\circ)} = \frac{\sin(-\delta_0)}{\sin(\epsilon)}$$
$$\sin(\delta_0) = -\sin(\theta_0)\sin(\epsilon)$$

Analogously:

$$\frac{\sin(\theta_{40})}{\sin(90^\circ)} = \frac{\sin(-\delta_{40})}{\sin(\epsilon)}$$
$$\sin(\delta_{40}) = -\sin(\theta_{40})\sin(\epsilon)$$ **1.0**

Combining the three previous expressions:

$$\sin(\delta_0) = \frac{\sin(\delta_{40})\cos(A_0)}{\cos(A_{40})}$$
$$-\sin(\theta_0)\sin(\epsilon) = \frac{-\sin(\theta_{40})\sin(\epsilon)\cos(A_0)}{\cos(A_{40})}$$
$$\sin(\theta_0) = \frac{\sin(\theta_{40})\cos(A_0)}{\cos(A_{40})}$$
$$\sin(\theta_0) = k\ \sin(\theta_{40})$$

2.0

Where $k = \dfrac{\cos(A_0)}{\cos(A_{40})} = \dfrac{\cos(120^\circ)}{\cos(98.5^\circ)} = 3.38$.
In order to replace one variable and solve for θ_0, it is possible to consider that the angular velocity of the Sun on the Ecliptic is approximately constant due to the Earth's very low eccentricity. In that case, the following expression must be true:

$$\theta_{40} = \theta_0 - \frac{40}{T_{year}} \times 360^\circ$$
$$\theta_{40} = \theta_0 - \beta$$

2.0

Where $\beta = \dfrac{40}{T_{year}} \times 360^\circ = 39.4^\circ$. Combining the expressions and solving for θ_0:

$$\sin(\theta_0) = k\ \sin(\theta_0 - \beta)$$
$$\sin(\theta_0) = k\ (\sin(\theta_0)\cos(\beta) - \sin(\beta)\cos(\theta_0))$$
$$\sin(\theta_0)(k\ \cos(\beta) - 1) = k\ \sin(\beta)\cos(\theta_0)$$
$$\therefore \tan(\theta_0) = \frac{k\ \sin(\beta)}{k\ \cos(\beta) - 1} = \frac{3.38\ \sin(39.4^\circ)}{3.38\ \cos(39.4^\circ) - 1}$$
$$\theta_0 = 53.1^\circ$$

3.0

Solving for δ_0:

$$\sin(\delta_0) = -\sin(\theta_0)\sin(\epsilon)$$
$$\delta_0 = -18.6^\circ$$

1.0

Solving for the latitude of the island:

$$\cos(\phi) = \frac{\sin(\delta_0)}{\cos(A_0)} = \frac{\sin(-18.6^\circ)}{\cos\left(180^\circ - \frac{120^\circ}{2}\right)}$$
$$\boxed{\phi = -50.4^\circ}$$

1.0

Note that only the negative solution to the equation above is relevant since it is known that the island is in the southern hemisphere.

(T8) Binary Hardening (25 points)

Consider a binary system of black holes with equal masses M separated by a distance a, revolving around their common center of mass (CM) in circular orbits. This binary moves against a very large, uniform field of stars (each of mass $m \ll M$) with number density n, interacting with them.

Consider a star that approaches the system from infinity with speed v and impact parameter b, in the reference frame of the CM (as shown in the figure below). Its closest approach distance to the CM is $r_p \approx \frac{1}{2}a$. For tasks (a) and (c), you should make use of the fact that $v^2 \ll \frac{GM}{a}$.

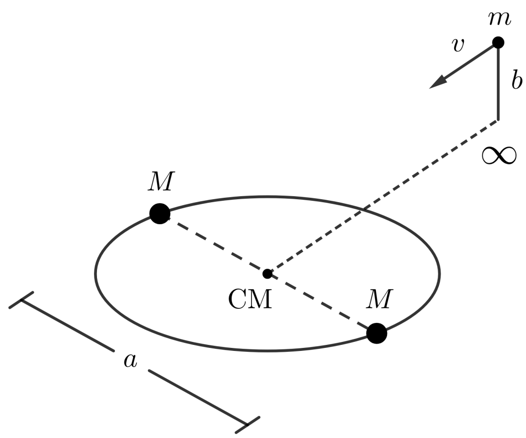


Figure 1: Diagram of the system

(a) (5 points) Obtain an expression for b, in terms of M, a, v, and physical constants. In this task, assume that the star interacts with the binary as if its total mass was fixed at the CM.

Solution: Let v_p be the closest approach velocity of the star. From conservation of mechanical energy:

$$-\frac{GM_T m}{r_p} + \frac{1}{2}mv_p^2 = \frac{1}{2}mv^2$$ **2.0**

where $M_T = 2M$ is the total mass of the binary. From conservation of angular momentum, we obtain v_p:

$$v_p = \frac{b}{r_p}v$$ **2.0**

setting $r_p \approx a/2$ and inserting v_p into the first equation, we solve for b:

$$b = a\sqrt{\frac{1}{4} + \frac{2GM}{av^2}}$$

$$\boxed{b \approx \frac{\sqrt{2GMa}}{v}}$$ **1.0**

Where we have used $v^2 \ll GM/a$ in the last step.

After a complex interaction with the binary, the star is slingshot from the system. The exact calculation of its ejection speed is complex, but the result can be estimated by considering that the star only interacts with one of the components when near the system. As such, consider, in part (b), *only* the gravitational interaction between the star and *one of the components* in the binary.

(b) (6 points) The star approaches the component with an initial speed negligible compared to the component's orbital speed, and both are moving directly towards each other. After interacting with the system, when the star is again far away from the black hole, we find that the direction of its velocity vector is inverted and the final speed is v_f. Determine v_f, in terms of M, a and physical constants. Assume that linear momentum and mechanical energy are conserved and that it takes place in a timescale much smaller than the binary's period. Recall that $m \ll M$.

Solution: The speed of the black hole in circular orbit can be found by equating the gravitational force to the centripetal net force:

$$\frac{GM^2}{a^2} = \frac{MV^2}{\frac{a}{2}}$$
$$\therefore V = \sqrt{\frac{GM}{2a}} \qquad \mathbf{1.0}$$

For the rest of the problem, we propose two different solutions:

Solution I:

Additionally, Let v_f be the ejection speed of the star, and V' the final speed of the black hole. To find v_f, we use conservation of linear momentum and conservation of mechanical energy for the two bodies:

$$MV = MV' + mv_f \qquad \mathbf{1.5}$$
$$\frac{1}{2}MV^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV'^2 \qquad \mathbf{1.5}$$
$$\frac{1}{2}MV^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}M\left(V - \frac{m}{M}v_f\right)^2$$
$$0 = v_f^2 - 2Vv_f + \frac{m}{M}v_f^2$$

Since $m \ll M$, we neglect the last term, yielding: **1.0**

$$\boxed{v_f = 2V = \sqrt{\frac{2GM}{a}}} \qquad \mathbf{1.0}$$

Because we are considering two moments where the distance between the star and the black hole is very large, we disregard potential energy terms.

Solution II:

In the reference frame of the component, the star approaches the component with a velocity $-\vec{V}$. **1.5**

And, since mechanical energy and linear momentum are conserved, it can be shown that relative speed between the components is the same before and after the interaction–that is, restitution coefficient equal to unity). Furthermore, since $M \gg m$, the velocity of the component can be considered unchanged (notice this is not valid in Solution I, as there we deal with second order terms arising from energy considerations) in the interaction. Therefore, after the interaction, the star is ejected back with a velocity $\vec{v}_f' = \vec{V}$ in the opposite direction. **1.5**

To return to the CM frame, we write $\vec{v}_f = \vec{v}_f' + \vec{V}$, such that

$$v_f = 2V \qquad \mathbf{2.0}$$

As we have previously found.

For the following task, assume that all stars approaching the system from infinity with an impact parameter $\frac{1}{2}b_0 \leqslant b \leqslant \frac{3}{2}b_0$ (where b_0 is the impact parameter of a star whose speed at infinity is v_0) attain a closest-approach distance $r_p \approx \frac{1}{2}a$ to the CM. **Also assume that all stars exit the system with the speed found in (b).**

(c) (14 points) Upon each encounter, part of the total energy of the binary is lost as kinetic energy acquired by a star. Assume that the binary orbit remains circular. Knowing this, using your results from previous tasks and taking into account *only* encounters with the stars within the specified range of impact parameters, show that the reciprocal of the binary's separation increases at a constant rate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = H\frac{G\rho}{v_0}$$

Here, $\rho = nm$ is the mass density of the star field, and G the universal gravitational constant. Find the dimensionless constant H, which refers to hardening.

Solution: First, we know the kinetic energy acquired by a star during an encounter:

$$\Delta K_* = \frac{1}{2}mv_f^2 - \frac{1}{2}mv^2 \approx \frac{GMm}{a}$$ **1.0**

Where we used that v^2 is negligible with respect to v_f^2, since $v_f^2 = \frac{2GM}{a}$, and $v^2 \ll GM/a$. This is also the energy lost by the binary during an encounter.

In order to find the rate of energy extraction, we must first obtain the rate of encounters, which can be estimated as follows:

imagine the binary travelling with velocity v in the reference frame of a far away star of impact parameter b. During a small interval $\mathrm{d}t$, all the stars included inside the annular cylinder of length $L = v\mathrm{d}t$ bounded by radii of b and $b + \mathrm{d}b$ will interact with the star. Call this number of stars $\mathrm{d}N$. It is given by:

$$\begin{aligned}
\mathrm{d}N &= n \cdot V_{cylinder} \\
&= n \cdot A_{annulus} \cdot L \\
&= n \cdot \pi \cdot [(b + \mathrm{d}b)^2 - b^2] \cdot v \cdot \mathrm{d}t \\
&\approx 2\pi n\, b\, v\, \mathrm{d}b\, \mathrm{d}t
\end{aligned}$$ **4.0**

Defining $\mathrm{d}\phi \equiv \frac{\mathrm{d}N}{\mathrm{d}t}$ to be the rate of encounters with stars of impact parameter between b and $b + \mathrm{d}b$ and using b from part (a),

$$\begin{aligned}
\mathrm{d}\phi &= 2\pi n\, b\, v\, \mathrm{d}b \\
&= 2\pi n \cdot \frac{\sqrt{2GMa}}{\cancel{v}} \cdot \cancel{v} \cdot \mathrm{d}b \\
\mathrm{d}\phi &= 2\pi n\sqrt{2GMa} \cdot \mathrm{d}b
\end{aligned}$$ **1.0**

To find the total rate of encounters ($\phi = \sum \mathrm{d}\phi$), we sum over all impact parameters in the interval $\frac{1}{2}b_0 \leqslant b \leqslant \frac{3}{2}b_0$. Notice $\sum \mathrm{d}b = \frac{3}{2}b_0 - \frac{1}{2}b_0 = b_0$, so that:

$$\begin{aligned}
\phi &= 2\pi n\sqrt{2GMa} \cdot \sum \mathrm{d}b \\
&= 2\pi n\sqrt{2GMa} \cdot \frac{\sqrt{2GMa}}{v_0} \\
&= \frac{4\pi nGMa}{v_0}
\end{aligned}$$ **3.0**

Since each encounter takes away ΔK_* and there are ϕ encounters per unit time, the rate of change of energy for the binary is:

$$\frac{\mathrm{d}E_{bin}}{\mathrm{d}t} = (-\Delta K_*) \cdot \phi = -\frac{4\pi n G^2 M^2 m}{v_0}$$ **2.0**

A change in the binary total energy can also be written as:

$$\frac{\mathrm{d}E_{bin}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{GM^2}{2a}\right) = -\frac{GM^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right)$$ **2.0**

Therefore. equating the two:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = \frac{8\pi G n m}{v_0}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = 8\pi\frac{G\rho}{v_0}$$ **1.0**

Thus, $\boxed{H = 8\pi}$

(T9) Physics of Accretion (35 points)

The accretion of matter onto compact objects, such as neutron stars and black holes, is one of the most efficient ways to produce radiant energy in astrophysical systems. Consider an element of gas of mass Δm in a stationary and geometrically thin disk of matter with a maximum radius of R_{max} and minimum stable orbital radius of R_{min} (with $R_{min}/R_{max} \ll 1$), in rotation around a compact object of mass M and radius R.

(a) (6 points) Assuming that an element of gas in the disk follows an approximately Keplerian circular orbit, find the expression of the total mechanical energy per unit of mass $\frac{\Delta E}{\Delta m}$ released by this gas from the moment it starts orbiting at a radius R_{max} until the moment it reaches an orbit of radius $r \ll R_{max}$. This process occurs very slowly, transforming kinetic energy into internal energy of the gas disk through viscous dissipation.
Note: Disregard the gravitational interaction between particles within the accretion disk and give your final answer in terms of G, M and r

Solution: For a Keplerian orbit:

$$\Delta m\frac{v_k^2}{r} = \frac{GM\Delta m}{r^2} \Rightarrow v_k^2 = \frac{GM}{r}$$ **2.0**

Total energy:

$$E = E_{kin} + E_{pot} \Rightarrow E(r) = \frac{1}{2}\Delta m v_k^2 - \frac{GM\Delta m}{r} \Rightarrow E(r) = -\frac{1}{2}\frac{GM\Delta m}{r}$$ **2.0**

$$E(R_{max}) - E(r) = \frac{1}{2}GM\Delta m\left(\frac{1}{r} - \frac{1}{R_{max}}\right) = \frac{GM\Delta m}{2r}\left(1 - \frac{r}{R_{max}}\right) \approx \frac{GM\Delta m}{2r}$$ **1.0**

Therefore:

$$\boxed{\frac{\Delta E}{\Delta m} \approx \frac{GM}{2r}}$$ **1.0**

(b) (5 points) Considering that the disk receives mass at an average rate of $\dot{M}$, and assuming that all the mechanical energy lost is ultimately converted into radiation, find an expression for the total luminosity of the disk.

Solution: Variation in energy:

$$\Delta E_{Tot} = E(R_{max}) - E(R_{min})$$

Luminosity:

$$L_{Tot} = \frac{\Delta E_{Tot}}{\Delta t} = \frac{\Delta E_{Tot}}{\Delta m} \times \dot{M}$$ **3.0**

$$\boxed{L_{Tot} = \frac{GM\dot{M}}{2R_{min}}}$$ **2.0**

Where the result from (a) was used, allied with $R_{min} \ll R_{max}$.

(c) (8 points) Consider now the ring composed of all mass elements from radius between $r + \Delta r$ to r. In this scenario, find an expression of the luminosity generated by the disc over its small length Δr at this radius, that is, find the expression for $\frac{\Delta E}{\Delta t \Delta r}$.

Solution: Variation in the energy of the tiny mass:

$$\Delta E = E(r + \Delta r) - E(r) = \frac{1}{2} GM\Delta m \left(\frac{1}{r} - \frac{1}{r + \Delta r}\right) \approx \frac{1}{2} GM\Delta m \left(\frac{\Delta r}{r^2(1 + \Delta r/r)}\right)$$

$$\Rightarrow \frac{\Delta E}{\Delta m} \approx \frac{1}{2} GM\Delta r \frac{1}{r^2}$$ **5.0**

Multiplying this expression by the rate of mass variation per time, it is possible to obtain an expression for $\frac{\Delta E}{\Delta t \Delta r}$:

$$\frac{\Delta E}{\Delta t} = \frac{\Delta E}{\Delta m} \times \dot{M} = \frac{GM\dot{M}\Delta r}{2r^2}$$

$$\Rightarrow \boxed{\frac{\Delta E}{\Delta t \Delta r} = \frac{GM\dot{M}}{2r^2}}$$ **3.0**

(d) (10 points) Assuming that the gravitational energy released in this ring is locally emitted by the surface of the ring in the form of blackbody radiation, find an expression for the surface temperature T of the ring.

Solution: Surface area of the ring:

$$A = 2[\pi(r+\Delta r)^2 - \pi r^2] = 2(\pi r^2 + 2\pi r\Delta r + \pi\Delta r^2 - \pi r^2)$$ **4.0**

$$\Rightarrow A \approx 4\pi r\Delta r$$

Using Stefan-Boltzmann's law:

$$A\sigma T^4 = L$$ **2.0**

$$4\pi r\Delta r\sigma T^4 = \frac{\Delta E}{\Delta t} \Rightarrow 4\pi r\sigma T^4 = \frac{\Delta E}{\Delta r\Delta t}$$ **2.0**

$$4\pi r\sigma T^4 = \frac{GM\dot{M}}{2r^2}$$

$$\Rightarrow \boxed{T = \left(\frac{GM\dot{M}}{8\pi\sigma r^3}\right)^{1/4}}$$ **2.0**

(e) (3 points) Consider that the central object is a stellar black hole with a mass of $3M_\odot$ and a rate of accretion of $\dot{M} = 10^{-9}\ M_\odot$/year. Consider also that $R_{min} = 3R_{sch}$, where R_{sch} is the Schwarzschild radius of the black hole. Determine the luminosity of the disk and the peak wavelength of emission of its innermost part. Disregard gravitational redshift effects and assume that the emission from the innermost part of the ring dominates the total emission.

Solution: Internal radius of the ring:

$$R_{sch} = \frac{2GM}{c^2} \Rightarrow R_{min} = 3R_{sch} = \frac{6GM}{c^2}$$

$$R_{min} = \frac{6GM}{c^2} = \frac{6\cdot 6.674\cdot 10^{-11}\cdot 3\cdot 1.988\cdot 10^{30}}{(2.998\cdot 10^8)^2} = 2.7\times 10^4\text{m}$$

Total luminosity:

$$L_{Tot} = \frac{GM\dot{M}}{2R_{min}} = \frac{\dot{M}c^2}{12}$$

$$\Rightarrow L_{Tot} = \frac{10^{-9}\cdot 1.988\cdot 10^{30}\cdot (2.998\cdot 10^8)^2}{12\cdot 365.2422\cdot 24\cdot 60\cdot 60}$$

$$\Rightarrow \boxed{L_{Tot} = 5\times 10^{29}\ W}$$ **1.5**

Surface temperature:

$$T = \left(\frac{GM\dot{M}}{8\pi\sigma R_{min}^3}\right)^{1/4} = \left(\frac{6.674 \cdot 10^{-11} \cdot 3 \cdot 10^{-9} \cdot (1.988 \cdot 10^{30})^2}{8\pi \cdot 5.670 \times 10^{-8} \cdot (2.7 \cdot 10^4)^3 \cdot 365.2422 \cdot 24 \cdot 60 \cdot 60}\right)^{1/4}$$

$$T = 5.5 \times 10^6\ K$$

Using Wien's law:

$$\lambda = \frac{b}{T} = \frac{2.898 \cdot 10^{-3}}{5.5 \cdot 10^6}$$

$$\Rightarrow \boxed{\lambda = 5 \times 10^{-10}\ m}$$ **1.5**

(f) (3 points) Now, considering another accretion system with $\dot{M} = 1\ M_\odot$/year and peak wavelength of emission is $\lambda = 6 \times 10^{-8}$ m, estimate the mass of this black hole.

Solution: Using Wien's law:

$$\lambda = \frac{b}{T} \Rightarrow T = \frac{b}{\lambda} = \frac{2.898 \cdot 10^{-3}}{6 \times 10^{-8}}$$

$$\Rightarrow T = 4.8 \cdot 10^4\ K$$ **1.0**

Considering this temperature to be that of the innermost of the accretion disk, and using $R_{\text{min}} = 3R_{\text{sch}}$:

$$T = \left(\frac{GM\dot{M}}{8\pi\sigma R_{min}^3}\right)^{1/4} \Rightarrow T^4 = \left(\frac{GM\dot{M}}{8\pi\sigma(3R_{\text{sch}})^3}\right) = \left(\frac{GM\dot{M}}{8\pi\sigma(6GM/c^2)^3}\right)$$

$$\Rightarrow M = \left(\frac{\dot{M}}{8\pi\sigma 6^3}\right)^{1/2} \frac{c^3}{GT^2}$$

$$\Rightarrow M = \frac{1}{(8\pi)^{1/2}} \frac{1}{6^{3/2}} \left\{\frac{1.988 \cdot 10^{30}/(3.15 \cdot 10^7)}{5.670 \cdot 10^{-8}}\right\}^{1/2} \frac{(2.998 \cdot 10^8)^3}{6.674 \cdot 10^{-11} \cdot (4.8 \cdot 10^4)^2}$$

$$\boxed{M \approx 2.5 \times 10^{39}\ kg \approx 1.3 \times 10^9\ M_\odot}$$ **2.0**

Long Questions

(T10) Greatest Eclipse (75 points)

The greatest eclipse is defined as the instant when the axis of the Moon's shadow cone gets closest to the center of the Earth in a solar eclipse. This problem explores the geometry of this phenomenon, using as an example the solar eclipse of May 29$^{\text{th}}$, 1919, which has a great historical significance for being the first time when astronomers were able to observationally verify general relativity. One of the scientific expeditions to observe this eclipse took place in the Brazilian city of Sobral.

The two following tables show the Cartesian and spherical coordinates of the Sun and the Moon at the time of the greatest eclipse. The system used for these coordinates is right-handed and has the origin at the center of the Earth, the positive x-axis pointing towards the Greenwich meridian, and the positive z-axis pointing towards the North Pole. For the rest of this problem, this will be referred to as **system I**.

Spherical coordinates:

	Center of the Sun	Center of the Moon
Radial Distance (r)	1.516×10^{11} m	3.589×10^{8} m
Polar Angle (θ)	$68°29'44.1''$	$68°47'41.6''$
Azimuthal Angle (φ)	$-1^h11^m28.2^s$	$-1^h11^m22.9^s$

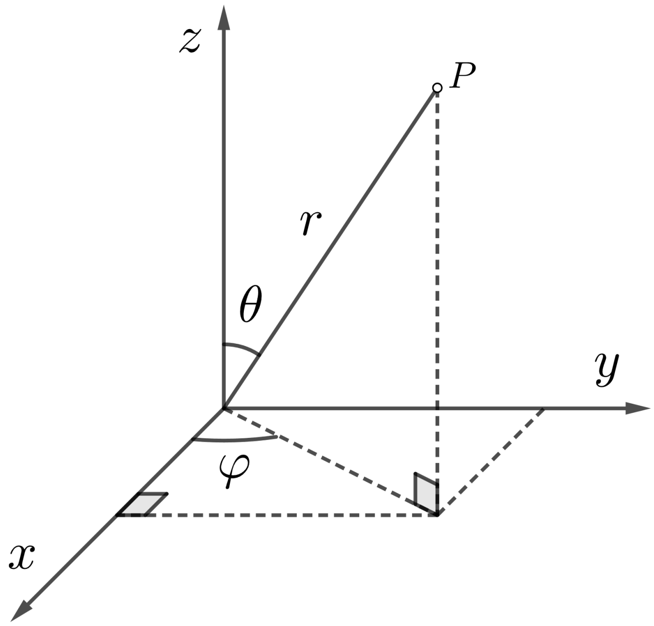
Cartesian coordinates:

	Center of the Sun	Center of the Moon
x	1.342×10^{11} m	3.185×10^{8} m
y	-4.327×10^{10} m	-1.025×10^{8} m
z	5.557×10^{10} m	1.298×10^{8} m

For this problem, assume that the Earth is a perfect sphere.

Note: The spherical coordinates of a point P are defined as follows:

- Radial distance (r): distance between the origin (O) and P (range: $r \geqslant 0$).
- Polar angle (θ): angle between the positive z-axis and the line segment OP (range: $0° \leqslant \theta \leqslant 180°$)
- Azimuthal angle (φ): angle between the positive x-axis and the projection of the line segment OP onto the xy-plane (range: $-12^h \leqslant \varphi < 12^h$)



Part I: Geographic Coordinates (25 points)

(a) (3 points) Determine the declination of the Sun and the Moon during the greatest eclipse for a geocentric observer.

> **Solution:** The declination is simply the complement of the polar angle:
>
> $$\delta = 90^\circ - \theta \quad \mathbf{1.0}$$
> $$\therefore \delta_\odot = 21^\circ 30' 15.9'' \quad \mathbf{1.0}$$
> $$\delta_{Moon} = 21^\circ 12' 18.4'' \quad \mathbf{1.0}$$

(b) (3 points) Determine the right ascension of the Sun and the Moon at the time of the greatest eclipse for a geocentric observer. The local sidereal time at Greenwich at that same moment was $5^h 32^m 35.5^s$.

> **Solution:** The local sidereal time at Greenwich corresponds by definition to the meridian of right ascension that is right above Greenwich. Since Greenwich corresponds to an azimuthal angle of 0^h, the right ascension at any given point corresponds simply to the sum of the local sidereal time at Greenwich and the azimuthal angle:
>
> $$\alpha = \varphi + LST_{Greenwich} \quad \mathbf{1.0}$$
> $$\therefore \alpha_\odot = 4^h 21^m 7.3^s \quad \mathbf{1.0}$$
> $$\alpha_{Moon} = 4^h 21^m 12.6^s \quad \mathbf{1.0}$$

(c) (4 points) Find a unit vector that indicates the direction of the axis of the Moon's shadow cone. This vector should point from the Moon to the vicinity of the center of the Earth.

Solution: By subtracting the Cartesian coordinates of the Sun from the Cartesian coordinates of the Moon, it is possible to obtain a vector for the direction of the axis of the Moon's shadow cone: **1.0**

$$\vec{\mathbf{u}} = < -1.339 \times 10^{11},\ 4.317 \times 10^{10},\ -5.544 \times 10^{10} > \text{m} \quad \mathbf{1.0}$$
$$|\vec{\mathbf{u}}| = 1.512 \times 10^{11}\,\text{m} \quad \mathbf{1.0}$$
$$\therefore \hat{\mathbf{u}} = < -0.8855,\ 0.2855,\ -0.3667 > \quad \mathbf{1.0}$$

(d) (15 points) Determine the latitude and the longitude of the point where the axis of the Moon's shadow cone crosses the surface of the Earth during the greatest eclipse.

Solution: If one draws a vector in the same direction as $\hat{\mathbf{u}}$ upto the earth's surface, then its magnitude can be found by the formula, where **M** is the position vector of the Moon, k is a constant, and $R_\oplus$ is the radius of the Earth:

$$|\vec{\mathbf{M}} + k\hat{\mathbf{u}}|^2 = R_\oplus^2 \quad \mathbf{5.0}$$

Solving for k:

$$\begin{aligned}
0 &= k^2(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) + 2k(\hat{\mathbf{u}} \cdot \vec{\mathbf{M}}) + \vec{\mathbf{M}} \cdot \vec{\mathbf{M}} - R_\oplus^2 \\
&= k^2 + 2k(\hat{\mathbf{u}} \cdot \vec{\mathbf{M}}) + |\vec{\mathbf{M}}|^2 - R_\oplus^2 \\
\text{Now, } |\vec{\mathbf{M}}|^2 - R_\oplus^2 &= 1.288 \times 10^{17}\,\text{m} \\
2(\hat{\mathbf{u}} \cdot \vec{\mathbf{M}}) &= -7.178 \times 10^{8}\,\text{m} \\
\therefore 0 &= k^2 - k(7.178 \times 10^8) + (1.288 \times 10^{17}) \\
k &= \frac{7.178 \pm \sqrt{(-7.178)^2 - 4 \times 12.88}}{2} \times 10^8
\end{aligned}$$

The vector will intersect the earth's surface at two points. Hence two solutions. However, only the first intersection with the sphere is relevant in this case, so only the smallest solution to the equation should be considered:

$$\begin{aligned}
k &= \frac{7.178 - \sqrt{(-7.178)^2 - 4 \times 12.88}}{2} \times 10^8 \\
&= 3.528 \times 10^8
\end{aligned} \quad \mathbf{4.0}$$

The vector of location of the greatest eclipse corresponds to the position vector of the Moon plus k times the direction vector $\hat{\mathbf{u}}$:

$$\begin{aligned}
\vec{\mathbf{p}} &= \vec{\mathbf{M}} + k\hat{\mathbf{u}} \\
&= < 6.091 \times 10^6,\ -1.828 \times 10^6,\ 4.854 \times 10^5 >
\end{aligned} \quad \mathbf{2.0}$$

It is possible to use the following expressions to convert this vector to the spherical system:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
&= 6.378 \times 10^6\,\mathrm{m} \\
\theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \quad \text{(valid for } z > 0\text{)} \\
&= 1.495\,\mathrm{rad} = 85^\circ 38' \\
\varphi &= \arctan\left(\frac{y}{x}\right) \quad \text{(valid for } x > 0\text{)} \\
&= -0.2915\,\mathrm{rad} = -16^\circ 42'
\end{aligned}
$$ **2.0**

The latitude corresponds to the complement of the angle θ, and the longitude corresponds to the angle φ:

$$\boxed{4^\circ 22'\mathrm{N},\ 16^\circ 42'\mathrm{W}}$$ **2.0**

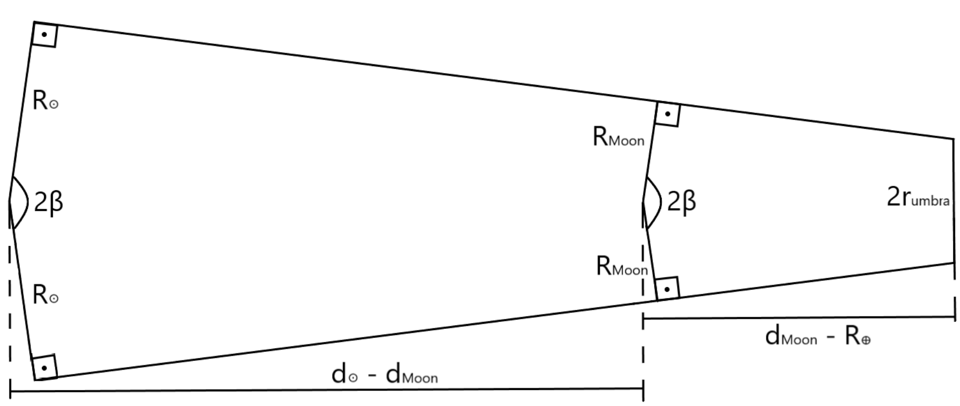
Part II: Duration of the Totality (50 points)

Precisely determining the duration of the totality of a solar eclipse involves complex calculations that would be beyond the scope of this problem. However, it is possible to obtain a reasonable approximation for this value using the two following assumptions:

- The size of the umbra on the surface of the Earth remains roughly constant throughout the totality for a given location.
- The velocity of the umbra on the surface of the Earth remains roughly constant throughout the totality for a given location.

(e) (10 points) Estimate the radius of the umbra during the greatest eclipse. In order to simplify the calculations, assume that the umbra is small enough that it can be considered approximately flat and that the axis of the Moon's shadow cone is extremely close to the center of the Earth during the greatest eclipse.

Solution: The approximate geometry described in the problem statement is shown below:



3.0

The angle β can be calculated as follows:

$$\begin{aligned}\beta &= \arccos\left(\frac{R_\odot - R_{Moon}}{d_\odot - d_{Moon}}\right)\\ &= \arccos\left(\frac{6.955\times10^8 - 1.737\times10^6}{1.516\times10^{11} - 3.589\times10^8}\right)\\ &= 89.74^\circ\end{aligned}$$ **3.0**

Using this angle, it is possible to determine the radius of the the umbra:

$$r_{umbra} = \frac{R_{Moon} - (d_{Moon} - R_\oplus)\cdot\cos(\beta)}{\sin(\beta)}$$ **3.0**

$$\boxed{r_{umbra} = 1.196\times10^5\,\mathrm{m}}$$ **1.0**

(f) (3 points) Calculate the velocity of the Earth's rotation at the latitude of the center of the umbra.

Solution: The Rotational velocity of the Earth at the latitude of the center of the umbra is the following:

$$\begin{aligned}v_{rot} &= \frac{2\pi R_\oplus}{23^h56^m04^s}\cos(\phi_{umbra})\\ &= \frac{2\pi\times6.378\times10^6}{23^h56^m04^s}\cos(4^\circ22')\end{aligned}$$ **2.0**

$$\boxed{v_{rot} = 463.7\,\mathrm{m/s}}$$ **1.0**

(g) (4 points) Determine the orbital velocity of the Moon at the instant of the greatest eclipse. Neglect the changes in the semi-major axis of the Moon's orbit.

Solution: Using the vis-viva equation:

$$v_{Moon} = \sqrt{GM_\oplus\left(\frac{2}{d_{Moon}} - \frac{1}{a_{Moon}}\right)}$$ **2.0**

$$= \sqrt{6.674\times10^{-11}\times5.972\times10^{24}\left(\frac{2}{3.589\times10^8} - \frac{1}{3.844\times10^8}\right)}$$

$$\boxed{v_{Moon} = 1088\,\mathrm{m/s}}$$ **2.0**

For the remaining items of this problem, assume that the tangential velocity of the Moon is roughly the same as the orbital velocity and redneglect its radial component.

In order to calculate the velocity of the umbra, it is convenient to define two new additional right-handed coordinate systems. **System II** is defined as follows:

- Origin (O_{II}): position of the Moon at the instant of the greatest eclipse.
- Positive x-axis: Tangent to the declination circle. Points eastwards.
- Positive y-axis: Tangent to the meridian of right ascension. Points northwards.

System III is defined as follows:

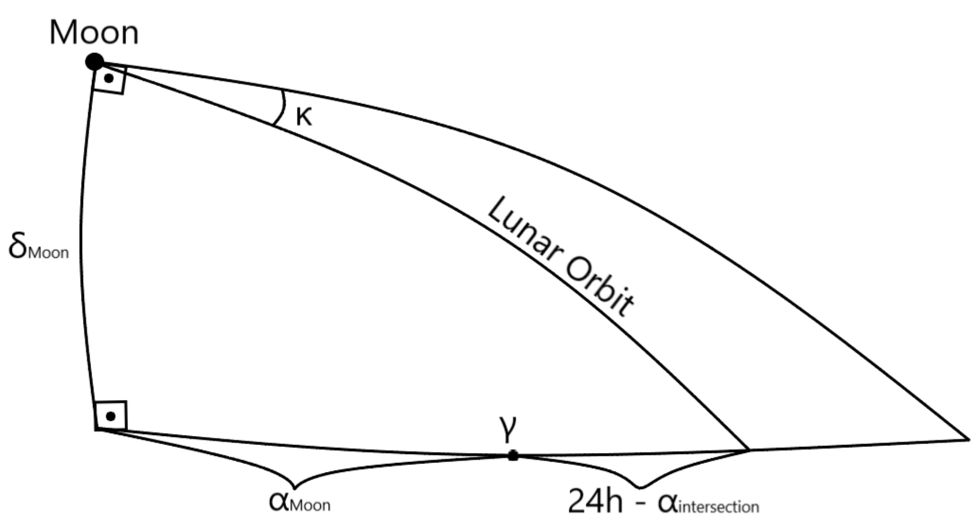
- Origin (O_{III}): center of the umbra at the instant of the greatest eclipse.
- Positive x-axis: Tangent to the latitude circle. Points eastwards.
- Positive y-axis: Tangent to the meridian of longitude. Points northwards.

Note that in both systems, the xy-plane is tangent to the celestial sphere at the position of the origin.

System III is similar to system II, with the only difference being that the origin (O_{III}) is at the center of the umbra at the moment of the greatest eclipse.

(h) (14 points) Using system II, determine the velocity vector of the Moon during the greatest eclipse. Note that the intersection between the Celestial Equator and the lunar orbit that is closer to the position of the eclipse has a right ascension of $23^h07^m59.2^s$.

Solution: It is possible to use the following diagram to determine the value of the angle κ between the x-axis and the velocity vector of the Moon:



6.0

Note that the great circle that intersects the Moon's meridian of right ascension at a right angle has a tangent line that coincides with the x-axis, so the angle κ on the figure is also the angle between the velocity vector of the Moon and the x-axis. Also note that the Moon is moving from west to east in the celestial sphere (to the left in the figure), so its velocity vector is on the first quadrant of the xy-plane on this system.

Using the four parts formula:

$$\begin{aligned}\cos(\delta_{Moon})\cos(90^\circ) &= \sin(\delta_{Moon})\cot(\alpha_{Moon} + 24^h - \alpha_{intersection}) \\ &\quad - \sin(90^\circ)\cot(90^\circ - \kappa) \\ \therefore 0 &= \sin(\delta_{Moon})\cot(\alpha_{Moon} - \alpha_{intersection}) - \tan(\kappa) \\ \tan(\kappa) &= \frac{\sin(\delta_{Moon})}{\tan(\alpha_{Moon} - \alpha_{intersection})} \\ \kappa &= \arctan\left[\frac{\sin(\delta_{Moon})}{\tan(\alpha_{Moon} - \alpha_{intersection})}\right] \\ &= \arctan\left[\frac{\sin(21^\circ 12' 18.4'')}{\tan(4^h 21^m 12.6^s - 23^h 07^m 59.2^s)}\right] \\ \kappa &= 4^\circ 17'\end{aligned}$$

3.0

2.0

Now it is possible to break down the velocity of the Moon into the x and y components:

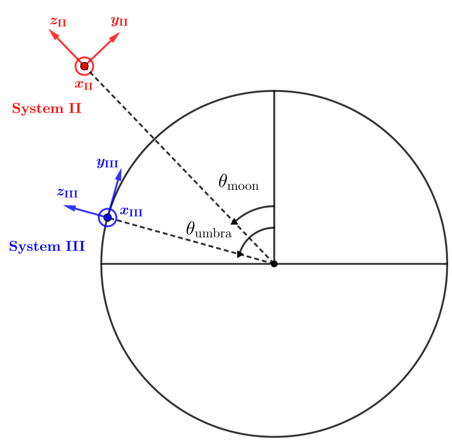
$$\mathbf{v_{II}} = \begin{bmatrix} v_{Moon}\cos(\kappa) \\ v_{Moon}\sin(\kappa) \\ 0 \end{bmatrix} = \begin{bmatrix} 1085\,\mathrm{m/s} \\ 81.25\,\mathrm{m/s} \\ 0 \end{bmatrix}$$ **3.0**

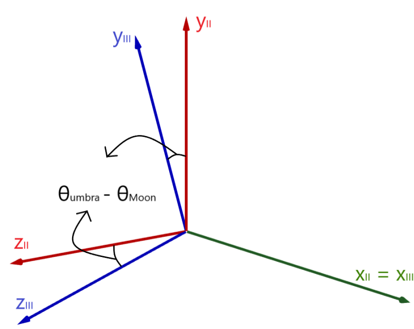
The radial component of the velocity was neglected, so the velocity on the z-axis is equal to zero.

(i) (10 points) Write the velocity vector of the Moon in system III. Note that in system I, the azimuthal angle difference between the positions of the origins O_{II} and O_{III} is negligible, so you should only take into account the difference in the polar angles.

Solution: It is possible to obtain system III by rotating system II from north to south by an angle of $\theta_{umbra} - \theta_{Moon}$ around the x-axis. The angles θ_{umbra} and θ_{Moon} correspond to the polar angles of the umbra and the Moon on system I.

The following two figures illustrate this rotation.





Since the rotation is around the x-axis, the x component remains unchanged. The component that was originally in the y-axis is broken down into two components on **3.0** the y-axis and the z-axis of the new system, which results in the following vector:

$$\mathbf{v_{III}} = \begin{bmatrix} v_{Moon} \cdot \cos(\kappa) \\ v_{Moon} \cdot \sin(\kappa) \cdot \cos(\theta_{umbra} - \theta_{Moon}) \\ -v_{Moon} \cdot \sin(\kappa) \cdot \sin(\theta_{umbra} - \theta_{Moon}) \end{bmatrix}$$ **6.0**

$$\mathbf{v_{III}} = \begin{bmatrix} 1085\,\text{m/s} \\ 77.76\,\text{m/s} \\ -23.54\,\text{m/s} \end{bmatrix}$$ **1.0**

(j) (6 points) Calculate the speed of the center of the umbra along the surface of the Earth at the instant of the greatest eclipse.

Solution: The velocity of the umbra corresponds to the vector sum of the component caused by the rotation of the Earth and the component caused by the velocity of the Moon.

Note that all points on the axis of the Moon's shadow cone have the same velocity, $\mathbf{v_{III}}$ is also the velocity of the axis of the Moon's shadow cone at the center of the umbra.

Since the umbra moves along the surface of the Earth, so the z_{III}-component should be set to zero.

$$\mathbf{v_{umbra}} = \begin{bmatrix} v_{III,x} - v_{rot} \\ v_{III,y} \\ 0 \end{bmatrix} = \begin{bmatrix} 1085 - 463.7 \\ 77.76 \\ 0 \end{bmatrix} = \begin{bmatrix} 621.4\,\text{m/s} \\ 77.59\,\text{m/s} \\ 0 \end{bmatrix}$$ **4.0**

The modulus of this vector is the following:

$$v_{umbra} = |\mathbf{v_{umbra}}| = \sqrt{621.4^2 + 77.76^2} = 626.3\,\text{m/s}$$ **2.0**

(k) (3 points) Estimate the duration of the totality of the eclipse at the location with the coordinates found on item (d).

Solution: Considering the assumptions listed at the beginning of part II, it is possible to estimate the length of the totality by dividing the diameter of the umbra by its speed:

$$\Delta t_{totality} = \frac{2 \cdot r_{umbra}}{v_{umbra}} = \frac{2 \times 1.196 \times 10^5}{626.3}$$ **2.0**

$$\boxed{\Delta t_{totality} = 6\text{min}21.4\text{s}}$$ **1.0**

(T11) Ground Tracks (75 points)

The projection of a satellite's orbit onto the Earth's surface is called its ground track. At a given instant, one can imagine a radial line drawn outward from the center of the Earth to the satellite. The intersection between the Earth's spherical surface and such radial line is a point on the ground track.

The location of this point is specified by its geocentric latitude and longitude. The ground track is then essentially the figure traced by this point as the satellite moves around the Earth.

Part I: Sun-Synchronous Orbits (25 points)

It is particularly interesting to analyze the ground track of a so-called Sun-Synchronous orbit, which is a nearly polar orbit around a planet where the satellite passes over any given point on the planet's surface at the same mean local solar time. This property is especially interesting for satellite imaging, ensuring similar illumination conditions over different days.

The figure below shows the ground track of a satellite in a Sun-Synchronous orbit. Its inclination angle (i) - the angle between the satellite orbital plane and the Earth's equatorial plane - falls within the range $90° < i < 180°$. The graph depicts five complete orbits of the satellite.

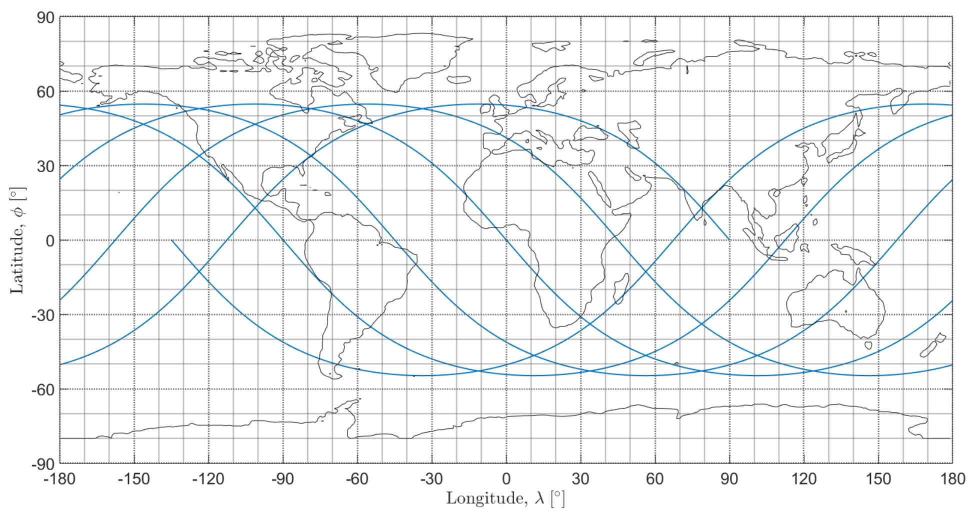


Figure 2: Ground Track for five orbits of the satellite

For the questions in Part I, assume that the Earth's orbit around the Sun to be circular.

(T11) (a) (3 points) Determine the nodal precession rate for a Sun-Synchronous orbit in rad/s.

> **Solution:** Since the orbit is Sun-Synchronous, this means that the satellite's line of nodes must drift 360° along one sidereal year. This ensures that the satellite will always pass over a given meridian at the same local mean solar time. Hence, **1.0**
>
> $$\dot{\Omega} = \frac{360°}{T_s} = \frac{2\pi}{365.2564 \cdot 24 \cdot 60 \cdot 60} = 1.991 \cdot 10^{-7} \ rad/s$$ **2.0**

(b) (8 points) Based on the ground track of Figure 2, determine the inclination of the satellite's orbit (in degrees) and estimate its orbital period (in minutes). Consider that the orbital period of the satellite is shorter than one sidereal day.

> **Solution:** After one orbit, the difference in longitude perceived by the satellite on the ground track is associated with the combined effects of the rotation of the Earth and

nodal precession. This is expressed by

$$\lambda_2 = \lambda_1 - \omega_\oplus \cdot T + \dot{\Omega} \cdot T$$ **2.0**

From the Ground Track, $\lambda_1 = 90°$ and $\lambda_2 \approx -135°$ for five orbits. Therefore, knowing the value of $\omega_\oplus$, the angular velocity of the rotation of our planet,

$$T = \frac{1}{5} \cdot \frac{\lambda_1 - \lambda_2}{\omega_\oplus - \dot{\Omega}} = \frac{1}{5} \cdot \frac{225° \cdot \pi/180°}{7.292 \cdot 10^{-5} - 1.991 \cdot 10^{-7}} \approx 10800\ s = 180\ min$$ **2.0**

Although $\dot{\Omega}$ is significantly smaller than $\omega_\oplus$, it is conceptually important to include the nodal precession rate in the formula.
The inclination of the orbit can be obtained directly from the Ground Track. Taking any of the two points of the orbit with the largest (absolute) value of latitude, one finds

$$|\phi_{max}| \approx 55°$$ **2.0**

Since $i > 90°$, $|\phi_{max}|$ is the supplementary angle of the inclination:

$$i = 180° - 55° = 125°$$ **2.0**

(c) (2 points) Estimate the semi-major axis a of the orbit in km.

Solution: Having calculated the period associated with the orbit, the semi-major axis is directly obtained from

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_\oplus} \implies a = \left(\frac{T^2 GM_\oplus}{4\pi^2}\right)^{1/3} = 1.06 \cdot 10^4\ km$$ **2.0**

(d) (1 point) Determine the number of orbits completed by the satellite until it returns to the same position on Earth.

Solution: Since the orbit is sun-synchronous, the satellite completes an integer number of orbits during one solar day. Hence,

$$N_S = \frac{24 \cdot 60}{180} = 8 \text{ orbits}$$ **1.0**

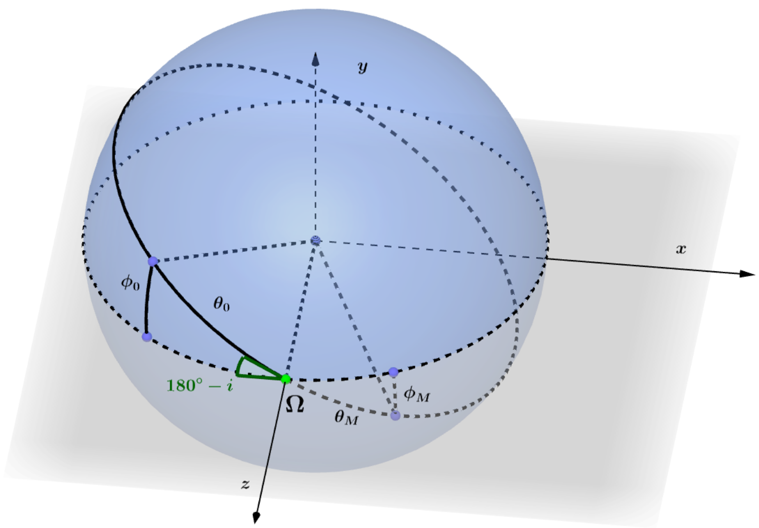
(e) (11 points) As it can be seen in Figure 2, the ground track crosses the Brazilian city of Maceió $(\phi, \lambda) = (9.7°S; 35.7°W)$ and also Chorzów $(\phi, \lambda) = (50.3°N; 19.0°E)$, in Poland. Knowing that the ground track crosses Maceió at noon (local time), determine the local time that the satellite track crosses Chorzów. Hint: specifically for this task, you may neglect the effects of nodal precession.

Solution: The satellite takes more than 2 orbits and less than 3 orbits to go from Poland to Brazil. Using as reference the point P_o ahead of Maceió that is exactly 3 orbits after Chorzów,

$$\lambda_o = \lambda_C - 3 \cdot 45^\circ$$

$$\phi_0 = \phi_C$$

Using sine law in the following spherical triangle,



$$\sin\theta = \frac{\sin\phi}{\sin(180^\circ - i)}$$ **2.0**

$$\theta_0 = \arcsin\left(\frac{\sin\phi_o}{\sin i}\right) = 69.9^\circ$$

$$\theta_M = \arcsin\left(\frac{\sin\phi_M}{\sin i}\right) = -12.0^\circ$$

The angle the satellite traveled will depend on how many complete orbits it took and the angle from the final whole orbit to Maceió:

$$\Delta\theta = n \cdot 360^\circ - (\theta_0 - \theta_M)$$ **3.0**

by the graph, $n = 3$, which results in $\Delta\theta = 998.1^\circ$ knowing its orbit is circular, **1.0**

$$\Delta t = T \cdot \frac{\Delta\theta}{360^\circ} = 8.32 \text{ h}$$ **1.0**

The local time at Chorzów in which the satellite passes by will be, considering $\lambda_M = 35.7^\circ W = -35.7^\circ$,

$$T_C = T_M - \Delta t + \omega_\oplus^{-1}(\lambda_C - \lambda_M)$$ **3.0**

$$\boxed{T_C \approx 7\text{h}20\text{min}}$$ **1.0**

Part II: Tundra orbits (50 points)

A Tundra orbit is a type of geosynchronous elliptical orbit characterized by a high inclination. The apogee is positioned over a specific geographic region, allowing for prolonged visibility and coverage over that area. This orbit ensures that a satellite spends the majority of its orbital period over the northern - or southern - hemisphere, making it particularly useful for communications and weather observation over high-latitude regions.

The image below represents the ground track of a satellite in a Tundra orbit with an argument of perigee equal to 270°. The satellite orbits the Earth in the same direction of its rotation. For the following items, you can disregard the effects of the Earth's oblateness.

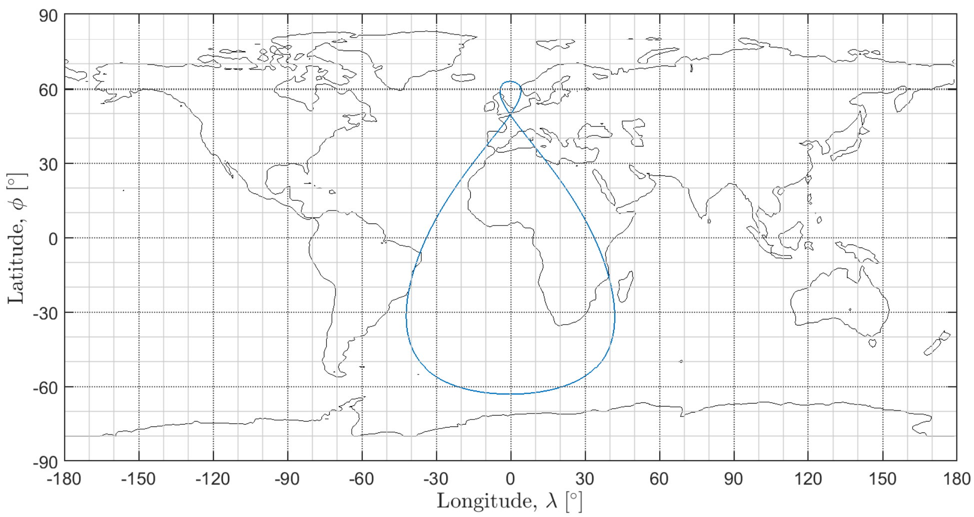


Figure 3: Tundra Orbit Ground Track for one orbital period

(f) (4 points) Based on the graph above, give the inclination of the satellite's orbit i (in degrees), its orbital period T (in minutes) as well as its semi-major axis a (in km).

Solution: For the inclination, we need to take the maximum latitude reached by the satellite, which gives us $\boxed{i = 63°}$. For the period, we have to realize that in one orbit, the Earth's rotation does not cause a shift in the orbit, so it must be geosynchronous, that is $T = T_{\oplus} = \boxed{1436 \text{ min}}$. For the semi-major axis, we apply Kepler's Third Law: **1.0** **1.0**

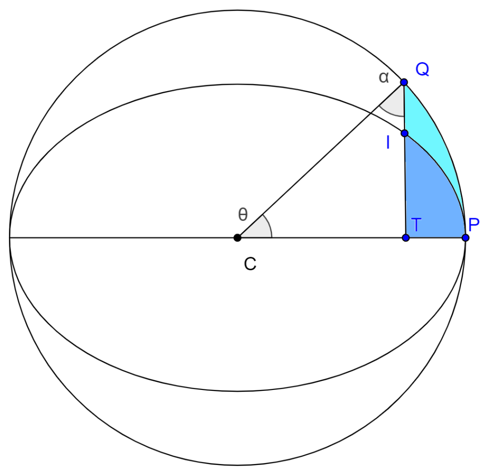
$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} \rightarrow \boxed{a = 42,164 \text{ km}}$$ **2.0**

(g) (12 points) Show that the time the satellite spends in the northern hemisphere is given by

$$T' = \left(\frac{1}{2} + \frac{\sin^{-1}(e)}{\pi} + \frac{e}{\pi} \cdot \sqrt{1 - e^2}\right) T$$

where e is the eccentricity of the orbit and T is its orbital period.

Solution: We will first find the expression for the area divided by the semi-latus rectum as a function of eccentricity so then we can apply the Kepler's Second Law to find the required time. To do this, we will use the projection of the area of a circle when it is inclined by an angle $\arccos\left(\frac{b}{a}\right)$, as shown in the figure below:



Thus, we will first calculate the hatched area of the circle figure A_{circle}

$$A_{circle} = A_{sector} - A_{triangle} = \frac{\theta}{2\pi} \cdot \pi a^2 - \frac{ae \cdot b}{2}$$ **4.0**

We can also use that $\theta = \frac{\pi}{2} - \alpha$ and $\alpha = \sin^{-1}(e)$. Therefore:

$$A_{circle} = \left(\frac{\pi \cdot a^2}{4} - \frac{\sin^{-1}(e) \cdot a^2}{2} - \frac{ab \cdot e}{2}\right)$$

Now, we project this area,

$$A_{ellipse} = A_{circle} \cdot \frac{b}{a}$$

$$A_{ellipse} = \left(\frac{\pi ab}{4} - \frac{\sin^{-1}(e) \cdot ab}{2} - \frac{b^2 e}{2}\right)$$ **4.0**

Thus, the area covered by the satellite's position vector from one semi-latus rectum to the other will be given by twice the area found above, as we can see from the image, so that the area on the perigee side is given by

$$A_{per} = \frac{\pi ab}{2} - b(a \sin^{-1}(e) + eb)$$

For the apogee side, it is enough to know that $A_{ap} = \pi ab - A_{per}$, where πab is the area of the whole ellipse. Therefore, we have:

$$A_{ap} = \frac{\pi ab}{2} + b\left(a \sin^{-1}(e) + eb\right)$$ **2.0**

Using Kepler's Second Law, we can write:

$$\frac{A_{ap}}{A_{ellipse}} = \frac{T'}{T}$$ **2.0**

Using that $A_{ellipse} = \pi ab$ and $b = a\sqrt{1-e^2}$, we have:

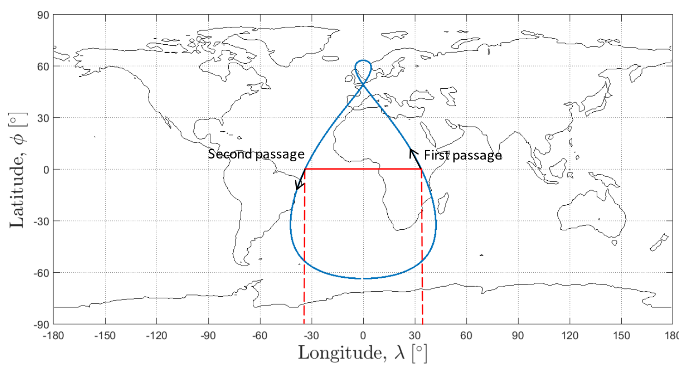
$$\boxed{\frac{T'}{T} = \frac{1}{2} + \frac{\sin^{-1}(e)}{\pi} + \frac{e}{\pi} \cdot \sqrt{1-e^2}}$$

(h) (10 points) Estimate numerically the eccentricity e of its orbit. You can consider that the eccentricity is so small that $\sin(e) \approx e$ and $e^2 \ll 1$.

Solution: Between the two consecutive passages through the semi-latus rectum, we can use the following equation:

$$\Delta\lambda = \pi - \omega_{\oplus} \cdot T'$$ **2.0**

Where $\Delta\lambda$ is the variation in longitude of the satellite between its passages through the semi-latus rectum, as shown in the image below.



Therefore, we can write using the expression of the last item

$$\Delta\lambda + \omega_{\oplus} \cdot T \cdot \left(\frac{1}{2} + \frac{\sin^{-1}(e)}{\pi} + \frac{e}{\pi} \cdot \sqrt{1-e^2}\right) = \pi$$

As $\omega_{\oplus} = \frac{2\pi}{T}$ (the orbit is geosynchronous), we will get:

$$\Delta\lambda + 2 \cdot \sin^{-1}(e) + 2e \cdot \sqrt{1-e^2} = 0$$

$$\sin^{-1}(e) + e \cdot \sqrt{1-e^2} = -\frac{\Delta\lambda}{2}$$ **2.0**

Considering that $\Delta\lambda \approx -67.5°$ from the graph (it is important to note that this value should be negative by the figure), we find from the above formula, by iteration, **2.0**

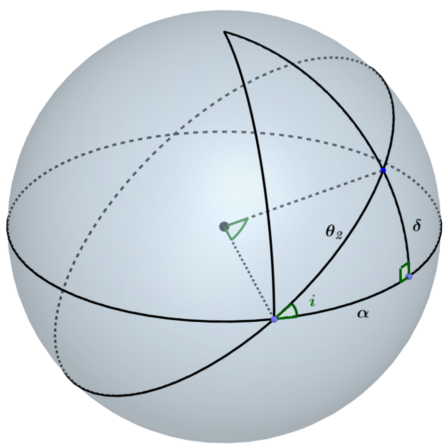
$\boxed{e \approx 0.3}$. If we use the approximations $\sin(e) \approx e$ and $e^2 \ll 1$ given in the statement, we find:

$$2e = -\frac{\Delta\lambda}{2}$$ **3.0**

which gives us an approximate answer $\boxed{e \approx 0.295 \approx 0.3}$ **1.0**

(i) (18 points) From the ground track, we can observe that the satellite exhibits a retrograde motion in both its northern and southern hemisphere trajectories. Find the true anomaly (in degrees) of the satellite at the beginning and end of its retrograde motion in the southern hemisphere.

Solution: At the start and end points of retrograde motion, we have $\omega_{RA} = \omega_\oplus$, where ω_{RA} is the right ascension angular velocity of the satellite. To find this velocity, we use the following spherical triangle: **2.0**



where $\theta_2 = \theta - 90°$, and it is known that:

$$\cos(\theta_2) = \cos(\delta)\cos(\alpha) \quad \text{(Cosine Law)}$$ **1.0**

$$\frac{\sin(\theta_2)}{\sin(\alpha)} = \frac{\cos(\delta)}{\cos(i)} \quad \text{(Sine Law)}$$ **1.0**

We substitute $\cos(\delta)$ from the first expression into the second expression, obtaining:

$$\frac{\sin(\theta_2)}{\sin(\alpha)} = \frac{\cos(\theta_2)}{\cos(\alpha)\cdot\cos(i)}$$

$$\tan(\theta_2)\cdot\cos(i) = \tan(\alpha)$$

We then derive the above expression with respect to time - it is possible to skip the derivative step by just decomposing the satellite's angular velocity vector.

$$\frac{\cos(i)}{\cos^2(\theta_2)}\dot{\theta} = \frac{\dot{\alpha}}{\cos^2(\alpha)}$$ **2.0**

Using the Cosine Law expression to replace $\cos(\alpha)$, we finally have:

$$\omega_{RA} = \dot{\alpha} = \frac{\cos(i)}{\cos^2(\delta)}\dot{\theta}$$ **2.0**

We can then calculate the value of $\dot{\theta}$ as a function of the distance of the satellite from the center of the Earth, using the conservation of angular momentum:

$$\dot{\theta} = \frac{2\pi}{T}\frac{a^2\sqrt{1-e^2}}{r^2}$$ **2.0**

We still have the polar expression for the distance r as a function of the polar radius, given by:

$$r = \frac{a(1-e^2)}{1+e\cdot\cos(\theta)}$$ **2.0**

Substituting everything into the expression that gives the condition for the start or end of retrograde motion, we find:

$$\frac{2\pi}{T}\frac{(1+e\cos\theta)^2}{(1-e^2)^{\frac{3}{2}}}\frac{\cos(i)}{\cos^2(\delta)} = \omega_{earth} = \frac{2\pi}{T}$$

$$\frac{(1+e\cos\theta)^2}{(1-e^2)^{\frac{3}{2}}}\frac{\cos(i)}{\cos^2(\delta)} = 1$$ **2.0**

From the graph, we find the latitude - or declination - of the start or end of retrograde motion, given by $\delta \approx -32°$. Thus, substituting all the variables into the above equation **2.0**
and solving for θ, we find: $\boxed{\theta = 54°}$ and $\boxed{\theta = 306°}$ as solutions. **2.0**

(j) (6 points) It is also noticeable that the ground track of a Tundra orbit has the shape of a figure-8, similar to an analemma, so that the satellite passes over the same point on Earth in a single orbit. Calculate the minimum eccentricity the orbit would need to have for this property to cease occurring. Use the same orbital inclination from the orbit in Figure 3.

Solution: For the figure-8 shape not to form, we need the inversion of motion in the northern hemisphere to occur exactly at the apogee position . Therefore, we will use **2.0**
exactly the same expression as the previous item, but with $\theta = 180°$ and eccentricity as the unknown variable . For the value of δ, as the satellite is at the position of maximum **1.0**
declination, $\delta = i$. We have:

$$\frac{(1-e)^2}{(1-e^2)^{\frac{3}{2}}}\frac{1}{cos(i)} = 1$$ **2.0**

Solving the above equation, we obtain: $\boxed{e \approx 0.4}$ **1.0**